dependent relations as there are involved unknown physical quantities, and to solve these equations simultaneously for the desired physical quantity. While the group of relations may consist of algebraic, differential, or integral equations, it is true that most physical behavior can be phrased in terms of differential equations; hence the need for thorough training in this branch of mathematics. After solution of the group of equations, it is the physicist's hope to interpret these results in terms of physical behavior. As an illustration of the points emphasized, the problem of the simple pendulum was carried through, showing how the free or natural motion of the pendulum may be oscillatory or non-oscillatory, depending on the frictional resistance encountered in swinging. The frequency of oscillation was shown to be independent of the mass of pendulum, of the amplitude of swing (approximately), and of the frictional resistance (approximately). The problem was made more general by considering the motion of one pendulum hung from another. Interpretation of the mathematical solution predicted beats between the forced and the natural oscillations of the lower pendulum, the phenomenon of resonance, and the relative phase of forced vibration and excitation. All of these conclusions drawn from mathematical treatment were experimentally demonstrated to be correct.

4. This paper takes account of the connection between the envelope of a system of integral curves and the singular solution of the corresponding ordinary differential equation. In particular it contains a brief discussion of the extraneous loci which may be found when seeking the envelope and a statement as to their distribution; also a short historical sketch.

5. This paper limits itself to a study of the positions of the six singular points of a three-cusped rational plane quintic curve. If the plane be divided into three sets of four triangles by the six lines determined by four singular points of which either one or three are cusps, then the two remaining singular points are in triangles of the same set.

6. Mr. L. Battig's paper treated of the two classes of plane quartic curves, rational and elliptic, that may be obtained by quadratic transformation of conics and plane cubics. With the exception of quartics with a triple point all rational quartics may be thus obtained from conics. Quartics with a triple point result from the transformation of singular cubics. Elliptic quartics (of deficiency one) result from the transformation of properly placed non-singular cubics (of deficiency one), since deficiency is unchanged under quadratic transformation.

LUCY T. DOUGHERTY, *Secretary*

## THE MARCH MEETING OF THE MICHIGAN SECTION

The seventh annual meeting of the Michigan Section of the Mathematical Association was held at the University of Michigan, Ann Arbor on March 22, 1930 in conjunction with the Michigan Academy of Science, Arts, and Letters.

Professor R. C. Shellenbarger, chairman, presided at the morning program and at the luncheon meeting. Professor Theodore Lindquist, chairman elect, presided at the afternoon program.

Fifty-seven persons registered, but there were others in attendance. Thirty-three members of the Association who attended the meeting are as follows: N. H. Anning, W. L. Ayres, J. W. Baldwin, W. D. Baten, Stanley Bolks, W. M. Borgman, Jr., J. W. Bradshaw, J. B. Brandeberry, Carl J. Coe, J. J. Corliss, S. E. Crowe, Albertus Darnell, Watson M. Davis, Lloyd C. Emmons, J. P. Everett, Peter Field, K. W. Folley, W. B. Ford, J. W. Glover, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, D. K. Kazarinoff, Theodore Lindquist, H. L. Olson, G. Y. Rainich, Louis J. Rouse, R. H. Schoonover, R. C. Shellenbarger, E. R. Sleight, G. G. Speeker, T. O. Walton, J. B. Winslow.

The following officers were elected for the ensuing year: Chairman, Theodore Lindquist, Michigan State Normal College; Secretary-treasurer, L. A. Hopkins, University of Michigan; Member of the Executive Committee, Clark L. Herron, Hillsdale College. Attention was called to the work of the two special committees which had been appointed in 1929 and whose reports were scheduled on the program:—first, the committee of which Professor L. C. Plant was chairman on the "Content of Mathematics Courses"; and second, the committee on "Legal Requirements of Mathematics Teachers" with Professor L. C. Emmons as chairman.

The following fifteen papers were read:

1. "On the motion of a top," by Mr. J. J. Corliss, University of Michigan.
2. "A note on cyclotomic equations," by Professor N. H. Anning, University of Michigan.
3. "Transformations with preservation of solid angles," by Professor G. Y. Rainich and Mr. E. H. Hildebrandt, University of Michigan.
4. "Content of mathematics courses," by Professor L. C. Plant, Michigan State College.
5. "On the normals to the parabola," by Mr. L. C. Mathewson, presented by Mr. Marvin Pahl, Albion College.
6. "Some generalizations of Pappus (Guldin's) theorems," by Mr. D. K. Kazarinoff, University of Michigan.
7. "On a certain system of differential equations," by Mr. G. C. Munro, University of Michigan.
8. "Combining certain probabilities which are constant," by Dr. W. D. Baten, University of Michigan.
9. "The evaluation of a certain generalized determinant," by Mr. W. O. Menge, University of Michigan,
10. "The Borel theorem and its generalizations," (The Chauvenet Prize Memoir) by Professor T. H. Hildebrandt, University of Michigan.
11. "A comparison of two expansions of the inverse sine function," by Professor A. L. Nelson, College of the City of Detroit.
12. "Constructional representation of numerical values involving complex

radicals, index 2, and its application in the inscribing or circumscribing of the general group of regular polygons," by Mr. C. E. Smith, Redford High School.

13. "Transformations of nets and surfaces," by Professor V. G. Grove, Michigan State College.

14. "Relation of mathematics to electrical engineering," by Professor J. H. Cannon, University of Michigan. (By special invitation).

15. "Legal requirements of mathematics teachers," by Professor L. C. Emmons, Michigan State College.

LOUIS A. HOPKINS, *Secretary*

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## THE NINETEENTH MEETING OF THE IOWA SECTION

The nineteenth meeting of the Iowa Section of the Mathematical Association of America was held with the Iowa Academy of Science at Iowa State College, Ames, Iowa, on May 2 and 3, 1930.

The attendance was about fifty, including the following twenty-eight members of the Association: F. A. Brandner, E. W. Chittenden, L. M. Coffin, Julia T. Colpitts, N. B. Conkwright, Marian E. Daniells, J. M. Earl, C. W. Emmons, C. H. Fischer, Annie W. Fleming, C. Gouwens, Gertrude A. Herr, E. C. Ingalls, F. M. McGaw, J. V. McKelvey, M. M. McKelvey, E. E. Moots, I. F. Neff, J. F. Reilly, H. L. Rietz, B. D. Roberts, Maria M. Roberts, E. R. Smith, G. W. Snedecor, J. S. Turner, L. E. Ward, C. W. Wester, Roscoe Woods.

The section chairman, Professor E. W. Chittenden, presided at both the Friday afternoon and Saturday morning sessions, relieved for a time by the Vice-chairman, Professor L. M. Coffin. Dinner was enjoyed together at the Memorial Union Friday evening and was followed by a short toast program.

At the business session officers for 1930–31 were elected as follows: Chairman, G. W. Snedecor, Iowa State College; Vice-chairman, E. C. Ingalls, Iowa Wesleyan College; Secretary-treasurer, J. F. Reilly, University of Iowa.

The necrology committee made the following report: "Since our last meeting, the Iowa Section of the Mathematical Association of America has lost through death two of its esteemed members—Professor E. A. Pattengill of Iowa State College, and Professor Fred Reusser of Buena Vista College. In recognition of their services as teachers of collegiate mathematics, their wholesome influence among students in their institutions, and their cooperation in furthering mathematical education, be it resolved that we express our deep sense of loss by a rising vote, and that this resolution be spread on the minutes and that the secretary send a copy to Mrs. Pattengill and to the nearest available relative of Mr. Reusser." H. L. Rietz, E. R. Smith, Committee.

A vote of thanks was tendered the department of mathematics at Iowa State College for the courtesies shown the section, and for the excellent arrangements made for our meeting.

The following recommendation was unanimously approved: "That a subject matter requirement be adopted for the certification of teachers of high school mathematics in Iowa, and that such subject matter requirement conform to that proposed by the North Central Association of Colleges and Secondary Schools."

The program consisted of eighteen papers, and an address by the retiring chairman, as follows:

1. "The distribution of the zeros for functions defined by a Heine series", by E. R. Smith, Iowa State College.

2. "Stencils in geometrical construction", by F. M. McGaw, Cornell College.

3. "Conformal and isohedral mapping of the earth on a plane", by Orlando C. Kreider, Iowa State College, by invitation.

4. "On the summation $\sum x^n$", by J. F. Reilly, University of Iowa.

5. "A lemniscate potential boundary value problem", by Archie Higdon, Iowa State College, by invitation.

6. "On certain appropriate scales in the graphic representation of frequency", by H. L. Rietz, University of Iowa.

7. "An algorithm for writing coefficients of a polynomial with known zeros," by C. W. Wester, Iowa State Teachers College.

8. "Progressive totals and summation as a substitute for accumulative multiplication", by A. E. Brandt, Iowa State College, by invitation.

9. "A note on the extraction of the square root on a calculating machine," by A. E. Brandt.

10. "Some theorems concerning parabolas having a common focus", by B. D. Roberts, Parsons College.

11. "Statistical control of a grading system," by G. W. Snedecor, Iowa State College.

12. "Some curves associated with two given curves", by C. H. Fischer, University of Iowa.

13. "The decomposition of $(x^p-1)/(x-1)$ for $p=41$, and $p=43$", by Cornelius Gouwens, Iowa State College.

14. "A representation of the law of mortality", by Arthur Ollivier, University of Iowa.

15. "The factors of $a^n-1$, $n\leqq 50$, $a=3, 5, 6, 7, 10, 11, 12$", by J. S. Turner, Iowa State College.

16. "Annuities certain for fractional periods", by J. F. Reilly, University of Iowa.

17. "The limaçon boundary value problem", by Daniel Hutton, Iowa State College, by invitation.

18. "An elementary proof of Descartes' rule of signs", by C. W. Wester, Iowa State Teachers College.

Abstracts of these papers follow:

1. Starting from results obtained by E. B. Van Vleck and Pringsheim, Professor Smith showed that the zeros of a function defined by a Heine series

are in any region finite in number. He also showed the possibility of defining a region about the origin which is entirely free of zeros.

2. Professor McGaw exhibited three stencils that he had made for the purpose of drawing accurate diagrams in his course in "Modern geometry". Stencil number one—the general triangle with about thirty points perforated, such as incenter, circumcenter, nine-point center, centroid, excenters, symmedians, Brocard points, etc. Stencil number two—the quadrilateral and quadrangle with all the necessary points for development of harmonic relations. Stencil number three—A general figure giving convenient cross-ratio forms for particular values.

3. In his discussion of Mercator's conformal projection Mr. Kreider found the ratio of the area of a portion of the sphere between two meridians to the corresponding area on the map in terms of the latitude. The paper was illustrated by several maps. Equal area projection was also discussed.

4. In his paper Professor Reilly emphasized the necessity of stating the final term in the usual formula given for the summation $\sum x^n$.

5. In his paper Mr. Higdon discussed a solution of the equation $\nabla^2\psi=0$ for the lemniscate by the use of definite integrals. The method involved the mapping of the region into a circle and obtaining the corresponding solution by means of a Poisson's integral which may be reduced to elliptic integrals of the first and second kinds.

6. In this paper Professor Rietz discussed the significance and usefulness of changes in scale in the graphic representation of certain frequency distributions. In particular special consideration was given to the general nature of distributions whose analysis is likely to be facilitated by the scales of arithmetic probability paper and of logarithmic probability paper.

7. Letting $s(r, k)$ represent the sum of the products $r$ at a time of the first $k$ numbers in the set $a_1, a_2, \cdots, a_n$, Professor Wester showed that $s(r, k) = s(r, k-1)+a_k s(r-1, k-1)$. This suggested arranging these sums, for purposes of calculation, in an array with $k$ for the number of the column and $r$ for the number of the row. The $k$th column then will contain the coefficients, after the first, of the polynomial whose zeros are the numbers $-a_1, -a_2, \cdots, -a_n$, and whose leading coefficient is unity.

8. A method was presented by Professor Brandt which enables one to eliminate all multiplication from problems involving the sums of squares or of products such as are met with in correlation studies and in inventories. This method has been developed for use with punched cards and sorting and tabulating machines. An explanation of the method step by step and an illustrative problem were given.

9. The fact that the sum of the first $n$ odd numbers is $n^2$ and that any number can be expressed in the form $a^2+(2a+b)b+R$ enables one to use division in extracting square roots. Any number of places can be determined by subtracting successive odd numbers and an equal number of places further may be secured by subtraction. The true root will be equal to or less than the

one extracted but can never be less by more than one in the last place. Professor Brandt showed that this method is especially useful in connection with calculating machines having automatic division. The method was also extended to the extraction of cube roots.

10. By means of the orthogonality of confocal parabolas Professor Roberts showed that two parabolas having a common focus and principal axes inclined at an angle $\alpha$ intersect at an angle $\alpha/2$, that if tangents be drawn to such parabolas at the points where a line from the focus intersects them, these tangents intersect at an angle $\alpha/2$, and that the locus of the intersection of such pairs of tangents is the "radical axis" of the parabolas.

11. The methods in vogue for controlling grading systems statistically are inadequate in this particular: they fail to make allowance for differences in ability among the groups to which the control applies. Such differences arise from (1) sectioning according to ability, (2) random sampling, and (3) differences in standards among departments and colleges in the same institution. Professor Snedecor showed that differences in ability among groups can be evaluated by using standard tests and the methods of multiple correlation. The result of an adequate control is that a given grade represents the same standard of attainment in every group of students in the institution.

12. In his paper Mr. Fischer suggested a general method of deriving curves from two given rational curves illustrated by the derivation of an ellipse from two concentric circles. The method consists of forming various combinations of the coordinates of two given rational curves, expressed in parametric form.

13. In Kraitchik's "Recherches sur la Théorie des Nombres" (1924) are displayed the polynomials $Y$ and $Z$ of the well known decomposition of $4X = 4(x^p-1)/(x-1)$ into the form $4X = Y^2-(-1)^{(p-1)/2}\ pZ^2$ for $p$ an odd prime. The largest value of $p$ there given is $p=37$. Legendre has given a method of obtaining these coefficients but it breaks down for $p=41$ and $p=43$. In this paper Professor Gouwens gave the polynomials $Y$ and $Z$ for these two values of $p$.

14. In his paper Mr. Ollivier reported on an attempt to represent a portion of the "American Experience Table of Mortality" by means of a parabola and a cosine series.

15. Professor Turner in his paper exhibited the factors of $a^n-1$, $n \leqq 50$, $a=3, 5, 6, 7, 10, 11, 12$, giving 48 results not found in Bickmore's paper (Messenger of Mathematics, vol. 25, p. 43).

16. In this paper Professor Reilly suggested a definition of the annuity symbols for fractional periods based upon interpolation with higher orders of differences.

17. Mr. Hutton gave a solution of the problem in potential theory where the contour is a limaçon.

18. Basing his proof on that given by L. E. Dickson in *First Course in Theory of Equations*, Professor Wester gave a shorter elementary proof of Des Cartes' rule of signs.

An abstract of Professor Chittenden's address on "General topology" follows:

Topology is concerned with the properties of sets of points which are invariant under bicontinuous one to one transformations. General topology applies the methods of the theory of abstract sets founded by Fréchet to the study of these questions. In this address the following topics were emphasized: the influence of the topological viewpoint on the development of the theory of abstract sets, the dimension theory of Urysohn and Menger, and the logical interrelations of the fundamental properties of abstract sets.

J. F. REILLY, *Secretary*

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## THE MAY MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at Carleton College, Northfield, Minn., on Saturday, May 17, 1930. Sessions were held at 11:00 o'clock and at 2:30 o'clock, with luncheon in the dining hall of Carleton College. Professor Fredrick Wood, chairman of the section, presided at both sessions.

Eighty-one persons attended the meeting, including the following twenty-four members of the Association: R. W. Brink, A. Bogard, W. H. Bussey, Elizabeth Carlson, June Constantine, H. H. Dalaker, Margaret Eide, Gladys Gibbens, C. H. Gingrich, W. L. Hart, Dunham Jackson, C. M. Jensen, W. H. Kirchner, W. H. McEwen, Marie Ness, M. F. Rosskopf, Inez Rundstrom, Edward Saibel, A. K. Solum, F. J. Taylor, Ella Thorp, Marion B. White, Gilbert Winklemann, Fredrick Wood.

Following the afternoon session those attending the meeting witnessed the beautiful May Fete given by the students of Carleton College.

During the afternoon session a vote of thanks was adopted in appreciation of the cordial hospitality of Carleton College and of the efforts of its department of mathematics. Officers for the following year were elected as follows: Chairman: C. H. Gingrich, Carleton College; Secretary, A. L. Underhill, University of Minnesota; Members of the Executive Committee: H. H. Dalaker, College of Engineering, University of Minnesota; Mrs. Margaret Eide, State Teachers College, River Falls, Wis.; A. Bogard, College of St. Teresa, Winona, Minn.

The following eight papers were presented:

1. "Some remarks on the *Rhind Mathematical Papyrus* and the *Source Book in Mathematics*", by Professor W. H. Bussey, University of Minnesota.

2. "A derivation of the equation for the normal surface", by Miss Marian Wilder, University of Minnesota.

3. "Some methods for the classical problems of geometry", by Professor Gladys Gibbens, University of Minnesota.

4. "Some qualities that should be possessed by a teacher of mathematics", the Chairman's address, by Professor Fredrick Wood, Hamline University.

5. "Mathematics as applied to biological problems", by John Stanley, University of Minnesota.

6. "A lemma in the theory of Fourier series", by Myron Rosskopf, University of Minnesota.

7. "The combination of dependent observations", by Phillip J. Roulon, University of Minnesota.

8. "A proof of H lder's inequality", by W. H. McEwen, University of Minnesota.

Abstracts of the papers follow:

1. Mr. Bussey spoke briefly of two new books recently published: "The Arnold Buffum Chace Edition of the Rhind Mathematical Papyrus" and "A Source Book in Mathematics" by David Eugene Smith. The purpose of his talk was to stress the desirability of having the two books available in every department of mathematics library for use by students and faculty.

2. The derivation of the equation of the normal surface can be based on the assumption of a frequency function $z = Cf(xy)$, where $f(xy) = e^{-ax^2-2bxy-cy^2}$. From the definitions of the correlation coefficient and the standard deviations, equations in $a$, $b$, $c$ are derived. The solutions of these equations, together with a determination of $C$, when substituted in the frequency function, give the standard form of the normal equation.

3. This paper will appear in an early number of the Monthly.

4. The paper suggested that a teacher should be a genuine student of the content and historic background of mathematics, and have a broad training in cultural subjects. He should love his subject and have imagination and intellectual curiosity. He should be interested in students, be gentle and kind, fair and square, and remain young enough to understand his students. Illustrations were given from various fields of mathematics to show the use of these qualities in stimulating the interest of the student and awaking him to the possibilities of doing original thinking.

5. The recent great advances made in physics by the use of mathematical reasoning lead to the question of whether or not living material lends itself to such treatment. It is possible to define several types of what may be called "biological entities". These are first, simple and compound, where the simple entity is a single organism, the compound, a group of organisms. Further division may be made into simple thinking, and simple non-thinking, compound with simple non-thinking elements, compound with simple thinking elements, thinking singly, and finally, compound with simple thinking elements thinking cooperatively and more or less in unison.

Examples of these entities are given and the feasibility of mathematical treatment is discussed in each case.

Certain analogies exist between these types. From a mechanistic viewpoint, the main characteristic of living matter is extreme heterogeneity of structure.

Under certain restrictions, the number of thoughts possible to a human be-

ing is finite, and small creatures are necessarily incapable of intricate thought.

Finally, the heterogeneous living system is oriented by means of a progressive extension, starting with the rigid motion, with all particles alike, passing en route through such systems as the non-rigid, compressible motion, with elements unlike, etc.

6. If $f(x)$ is a continuous function in the interval $(-\pi, \pi)$ with the period $2\pi$ and if $\int_{-\pi}^{\pi} f(x) \cos nx\, dx = 0$ and $\int_{-\pi}^{\pi} f(x) \sin nx\, dx = 0$ for all $n = 0, 1, 2, \cdots$, it follows that $f(x) \equiv 0$. The lemma allows a very concise proof of the $\delta, \epsilon$ part of the proof of convergence in Fourier series.

7. The problem of this paper is suggested by the situation which arises when a series of tests are applied to an experimental group and to a control group of students in order to test the efficacy of some educative process. In case the tests are independent, one can sometimes infer a significant difference between the groups even when the difference for each test is not significant. The paper points out the danger of applying the method when the tests are correlated; and it suggests methods of getting the best value from a set of correlated observations, and of measuring its significance. These methods will be developed in a later paper.

8. This paper offers a proof of the relation

$$\int_a^b \phi dx \leqq (b-a)^{(r-1)/r} \left\{ \int_a^b \phi^r dx \right\}^{1/r},$$

where $\phi > 0$ and $r$ is any real number $\leqq 1$. It is first shown by means of Schwarz's inequality that

$$\int_a^b \phi^i dx < (b-a)^{1/(i+1)} \left\{ \int_a^b \phi^{i+1} dx \right\}^{1/(i+1)}, \qquad i = 1, 2, \cdots, (n-1),$$

and by combining these $n-1$ relations, that the desired inequality holds for any positive integral value $n$. This fact is used to prove that

$$\int_a^b \psi^p dx \leqq (b-a)^{1-p} \left\{ \int_a^b \psi dx \right\}^p,$$

where $p$ is any real number $<1$, and the proof is then completed by identifying $\phi^r$ with $\psi$ and $1/r$ with $p$.

R. W. BRINK, *Acting Secretary*

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## THE MAY MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The twenty-seventh regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at Goucher College, Baltimore, Md., on Saturday, May 10, 1930. Sessions were

held in the morning and in the afternoon; Professor W. F. Shenton, Chairman of the Section, presided at both sessions.

Fifty-four persons attended the meeting, including the following forty-five members of the Association: O. S. Adams, Beatrice Aitchison, G. F. Alrich, R. N. Ashmun, H. G. Avers, Clara L. Bacon, W. J. Berry, G. A. Bingley, C. C. Bramble, J. L. Clayton, Teresa Cohen, A. Cohen, G. R. Clements, Tobias Dantzig, Alexander Dillingham, J. A. Duerksen, J. B. Eppes, Mary Ewin, P. J. Federico, Michael Goldberg, W. M. Hamilton, F. E. Johnston, H. P. Kaufman, L. M. Kells, W. D. Lambert, C. L. Leiper, Florence P. Lewis, Florence M. Mears, F. D. Murnaghan, C. H. Rawlins, Jr., Lowell Reid, W. F. Reynolds, A. W. Richeson, H. M. Robert, Jr., H. A. Robinson, R. E. Root, W. F. Shenton, T. H. Taliaferro, J. H. Taylor, Marian M. Torrey, John Tyler, P. Wernicke, J. Williamson, R. C. Yates, O. Zariski.

The following officers were elected: *Chairman*, Clara L. Bacon, Goucher College; *Secretary*, Edgar W. Woolard, The George Washington University; additional members of the executive committee, J. Williamson, The Johns Hopkins University, and G. A. Bingley, St. Johns College.

During a discussion which followed the program, encouragement was given to the preparation of papers appropriate to the general membership of the Association—papers more in line with the teaching membership than the graduate membership. The executive committee was asked to make an early effort to arrange programs for the meetings, and to provide for the entertainment of the group when no formal invitation was received from some institution.

An invitation from the chairman to hold the next meeting of the Section at American University was accepted.

The following five papers were presented:

1. "On linear subspaces of a given space," by Dr. Zariski, Johns Hopkins University.

2. "Mathematical aspects of a theory of the frequency distribution of species," by Florence P. Lewis, Goucher College.

3. "The potential of a spherical zone," by W. D. Lambert, U. S. Coast and Geodetic Survey.

4. "Some episodes from the history of the infinite," by Tobias Dantzig, University of Maryland.

5. "On a problem of regions," by H. A. Robinson, Johns Hopkins University.

Mr. O. S. Adams also presented a brief statement of several of his recent problems in the American Mathematical Monthly.

An abstract of one of these papers follows:

3. Mr. Lambert's paper dealt with the Newtonian potential of a very thin layer of attracting matter on the surface of a sphere and bounded by a circle of any size, the attracted point being also on the surface. This problem has important geophysical applications. Woodward ("On the form and position of sea level," U. S. Geological Survey Bulletin No. 48, Washington, 1888) showed

that the problem could be reduced to the evaluation of an elliptic integral, but did not actually use this method. It turns out to be a complete elliptic integral of the third kind and in this paper three methods of evaluating it numerically are given. A method is also given for computing the potential of any layer of constant surface density bounded by arcs of great circles, the attracted point being on the surface. The calculation likewise involves complete elliptic integrals of the third kind. In both problems the key to the solution is the suitable choice of the variable of integration. It is expected that these developments will be incorporated in a publication of the U. S. Coast and Geodetic Survey.

EDGAR W. WOOLARD, *Secretary*

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## SOME CONSTRUCTIONS FOR THE CLASSICAL PROBLEMS OF GEOMETRY

By GLADYS GIBBENS, University of Minnesota

The purpose of the following paper is to give a brief outline of some of the methods used by Mr. Gottfried Lenzer of St. Paul, Minnesota for handling the classical construction problems of geometry: the squaring of a circle, the duplication (or halving) of a cube, and the trisection of an angle. In March, 1928, the University of Minnesota was notified that Mr. Lenzer had bequeathed to the University a series of copyrighted drawings and explanatory notes concerning these problems, and it became my interesting task to go through the bequest, reorganize the work, and publish the results. While Mr. Lenzer was apparently not acquainted with the literature on the subject, he was not a circle-squarer in the sense of one who believed he had accomplished the impossible, but rather a man who found interest and companionship during many years of a solitary existence in the study of the subject and the execution of the constructions he had formulated. Although his results do not constitute any material addition to knowledge in the present state of things, they are of interest in themselves as well as for their revelation of the mind of a man who regarded them as his most cherished possession but who made no attempt to publish them during his lifetime.

Mr. Gottfried Lenzer was a native of Germany who lived in St. Paul for many years, and who was an engineer by profession. I do not know how his interest in these problems was originally awakened, but to judge by the dates on the drawings it was of long standing, and he continually reworked the constructions and developed his methods, trying to inter-relate the problems. His bequest consists of some sixty drawings, dating from 1911 to 1927, when the entire set was copyrighted, and a scant set of explanatory notes. The drawings are all on large paper, and are beautifully executed in colored inks. They are all very complicated, but are symmetric with respect to all convenient lines and points, an obvious esthetic pleasure to the engineer-draughtsman. Mr. Len-

zer's achievement seems quite remarkable for a man who was not mathematically trained, and who at no time attempted to formulate a general method for his constructions, but who handled each special case of the problem under consideration on its own merits; however he must have had some such general method in mind and in the following paper I have attempted to formulate these methods from the drawings and notes.

I. *The Trisection of an Angle: A Solution.*

Place the angle $AOA'$ (Figure 1) that is to be bisected so that its internal bisector is the vertical line $Y'OY$ and its external bisector the horizontal line

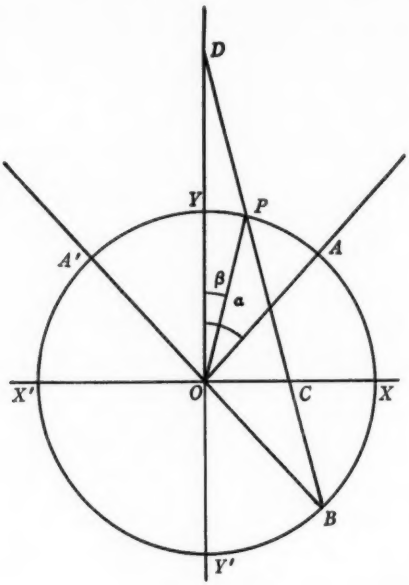


FIG. 1

$X'OX$. Draw a convenient circle having the vertex $O$ as center and meeting the sides of the given angle at $A'$ and $A$. Extend the side $A'O$ of the given angle to meet this circle again at $B$. Now mark off on a straight edge two radii, $CP$, $PD$ of the circle, and shift the straight edge through $B$ until $C$ lies on $OX$, $P$ on the circle, and $D$ on $OY$. It is obvious that if $CP=OP$, then $PD=OP$, and the three conditions can be fulfilled. (It is at this point that the construction ceases to be one by ruler and compass in the classical sense; we are using a marked ruler, and what might be termed three-point contact with a given figure.) Draw $OP$; it is a trisecting radius and its symmetric $OP'$ with respect to $OY$ is the other. Proof. Let $\alpha=\angle AOD=\frac{1}{2}\angle AOA'$, $\beta=\angle POD=\angle PDO$;

then $\angle OPC = 2\beta = \angle OBP$; $\angle OCP = 90° - \beta$; $\angle BOC = 90° - \alpha$ and $\angle BCO = 90° + \beta$. But in the triangle $OCB$, the sum of the three angles is 180°, so that $(90° - \alpha) + (90° + \beta) + 2\beta = 180°$, or $\beta = \frac{1}{3}\alpha$. That is, $OP$ is a trisecting radius for the half-angle $AOD$ and hence for the original angle $AOA'$.

If we wish to trisect the remaining arc of the circle $AXY'X'A'$, we shift the marked rule in similar fashion through $B$ so that $C$ is on $OX$, $P$ on the circle and $D$ on $OY'$. The proof goes through in the same way.

## 2. *Squaring the Circle: An Approximation.*

Mr. Lenzer's drawings are symmetric with respect to the horizontal and vertical diameters of the circle, but in order to avoid confusion of notation I shall give only one of the constructions involved; the remaining ones are obvious. Let the circle with center $O$ (Figure 2) be the circle to be squared; for

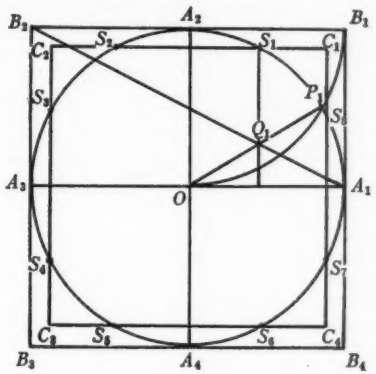


FIG. 2

convenience consider its radius as one unit. Circumscribe about it the square $B_1B_2B_3B_4$ with horizontal and vertical sides, labeling the points of tangency $A_1$, $A_2$, $A_3$, $A_4$ as indicated. With $A_2$ as center, draw a quarter unit-circle $OB_1$, meeting the original circle at $P_1$; draw $OP_1$ and $B_2A_1$ meeting at $Q_1$. At $Q_1$ erect a vertical line meeting the circle at $S_1$. This same construction can be made using $A_2$ for the center of the circle and $B_1A_3$ for the diagonal of the half square; then seven other points $S$ symmetric to $S_1$ with respect to $A_3A_1$, $A_4A_2$ and the diagonals of the square will be obtained. The square $C_1C_2C_3C_4$ through these eight points has an area approximately equal to that of the given circle.

*Closeness of the approximation.* If we set up a rectangular coordinate system with $A_3A_1$ and $A_4A_2$ as $x$- and $y$-axes respectively, then the equation of $OP_1$ is $y\sqrt{3} = x$ and of $B_2A_1$ is $x + 2y = 1$; they meet at $Q_1$ whose coordinates are $(x = 2\sqrt{3} - 3,\ y = 2 - \sqrt{3})$. The vertical line $Q_1S_1$, $x = 2\sqrt{3} - 3$, meets the given circle $x^2 + y^2 = 1$ at $S_1$ whose $y$-coordinate is given by $y^2 = 12\sqrt{3} - 20$. The

area of the square $C_1C_2C_3C_4$ having $2y$ as its side is $4y^2=4[12\sqrt{3}-20]=3.1384$. The area of the original circle is 3.1416. The approximating square then is too small; its area being .999 times that of the circle. But this error is well within the limits of ordinary drawing; if the original circle has a radius of 10 inches, the side of the approximating square is too short by less than .004 inches, less than the ordinary pen line.

*The square whose perimeter is approximately equal to the circumference of the given circle.* Inscribe a circle in $C_1C_2C_3C_4$, meeting $OS_k$ at $T_k$, $(k=1, 2, \cdots, 8)$ Then the square $D_1D_2D_3D_4$ through these points $T_k$ has a perimeter approximately equal to the circumference of the original circle. The proof is analogous to the one just given, and the error is of the same order.

The drawings also include constructions for circling the square, but since it is essentially the inverse of the preceding, no details will be given.

3. *Halving the Cube: An approximation.*

Let $A_1A_2A_3A_4$ (Figure 3) represent one face of the given cube. Divide the half-diagonal $OA_1$ into four equal parts by the points $C_1$, $D_1$, $E_1$. Draw a circle with $C_1$ as center and $C_1A_1$ as radius, and a line through $C_1$ parallel to $A_4A_1$.

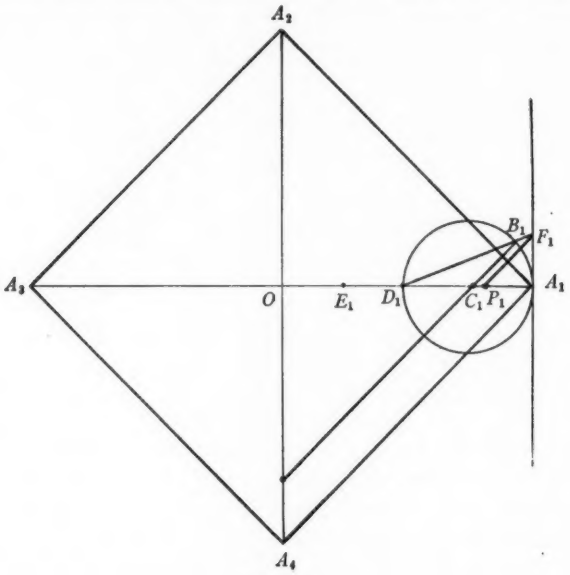


FIG. 3

Let $B_1$ be their point of intersection. Draw $D_1B_1$, extending it to meet a parallel to $OA_2$ through $A_1$ at $F_1$. Now through $F_1$ draw a parallel to $C_1B_1$, meeting $OA_1$ at $P_1$. Then $OP_1$ is a half-diagonal of one face of the required cube whose volume is approximately equal to half the volume of the original cube.

*Analytic discussion of the degree of the approximation.* Let us take the half-diagonal of the given cube as 4; then the length of an edge of the cube is $4\sqrt{2}$ and its volume is $128\sqrt{2}$. The equation of the circle $C_1$ is $(x-3)^2+y^2=1$; it meets the line $C_1B_1$ whose equation is $y=x-3$ at the point $B_1:(3+\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$. The equation of $B_1D_1$ is $x-y(\sqrt{2}+1)-2=0$, meeting $A_1F_1$ or $x=4$ at $(4, 2\sqrt{2}-2)$. Then the equation of $F_1P_1$ is $y-2\sqrt{2}+2=x-4$, so that the coordinates of $P_1$ are $(6-2\sqrt{2}, 0)$. The edge of the cube having $P_1P_2P_3P_4$ as one face (where $P_2,P_3,P_4$ are the corresponding points to $P_1$ on the other diagonals) is $6\sqrt{2}-4$ and the volume of the cube then is $8(3\sqrt{2}-2)^3=16[45\sqrt{2}-58]$. But the volume of a cube just half the original is $64\sqrt{2}$, so the approximating cube is smaller than the desired one, with an error of .3%. If the edge of the original cube were 10 inches, the width of the strip between the sides of the approximating cube and the true half-cube would be less than .002 inches, well within the limits of drawing.

In a single sentence in the notes, Mr. Lenzer states that an analogous method could be used for duplicating a cube, but none of the drawings show such a construction.

4. *Construction of a Regular Pentagon.*

Draw a circle with $OA_3$ as radius (Figure 4) and a concentric circle whose radius is twice as large. Draw the horizontal and vertical diameters as indicated.

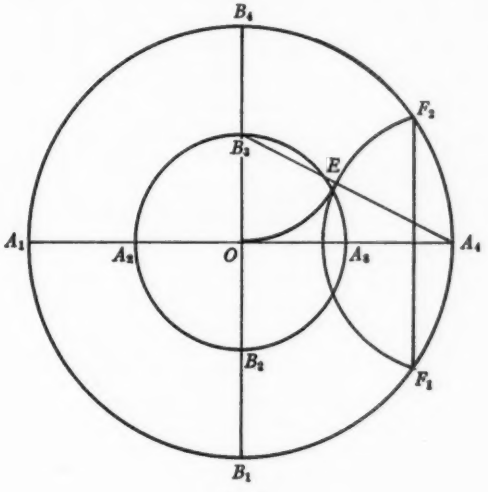


FIG. 4

Draw $B_3A_4$, meeting a circle with $B_3$ as center and radius $OB_3$ at $E$. Draw a circle with $A_4$ as center and radius $A_4E$, meeting the large circle at $F_1$ and $F_2$. Then $F_1F_2$ is a side of a regular pentagon inscribed in the circle $A_1B_4A_4B_1$.

*Analytic Proof.* Let us take $A_1A_4$ and $B_1B_4$ as $x$- and $y$-axes respectively, and $OA_4$ as a unit. Then the equation of $B_3A_4$ is $x+2y=1$; the equation of the circle with center at $B_3$ is $x^2+(y-\frac{1}{2})^2=\frac{1}{4}$; and the coordinates of $E$ are $(\frac{1}{5}\sqrt{5}, \frac{1}{10}(5-\sqrt{5}))$. The length $(EA_4)$ is $\frac{1}{2}(\sqrt{5}-1)$ and the equation of the circle with center at $A_4$ is $(x-1)^2+y^2=\frac{1}{2}(\sqrt{5}-1)$. It meets the original circle $x^2+y^2=1$ at $F_1$ and $F_2$ whose coordinates are

$$x = \tfrac{1}{4}(1+\sqrt{5}) = \cos 36^\circ, \quad y = \mp \tfrac{1}{4}\sqrt{(10-2\sqrt{5})}.$$

so that $F_1F_2$ subtends at 0 an angle of 72° and is a side of the inscribed pentagon.

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## ON OVALS

By N. HANSEN BALL, Massachusetts Institute of Technology

1. *Introduction.* During recent years many geometers have shown increased interest in those properties of plane curves which do not depend upon their analytical expression. It is surprising how many such properties have already been discovered, although the field has been barely touched. It is the purpose of this note to give a brief review of the most important publications on this subject. While some of these properties hold for any type of curve whatever most of them are valid only for curves which are closed and convex, i.e., which have at most two intersections with any line in the plane. Theorems of the latter type only shall be dealt with here.

2. *The four-vertex and allied theorems.* The four-vertex theorem states that on every closed, convex curve, or oval, there are at least four points where the radius of curvature has an extreme value. In this and other theorems of the same type, the further assumption is made that the radius of curvature is continuous for each point and is never zero. Since an ellipse has exactly four such points, if it can be established that no oval can have less than four, the minimum number is shown to exist and is definitely determined. In 1912, A. Kneser[1] gave a purely geometrical proof of this theorem. Shortly afterward W. Blaschke[2] gave three different proofs for it. One was based on the application of Fourier series and another, due to G. Herglotz, made use only of elementary differential geometry. The third, and most elegant, proof depends only on the equations, valid for any closed curve,

$$(1) \qquad \oint \rho(\phi) \sin \phi d\phi = 0 \text{ and } \oint \rho(\phi) \cos \phi d\phi = 0,$$

where $\phi$ is the angle between the tangent to the curve and a fixed line in the

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[1] A. Kneser, *Bemerkungen über die Anzahl der Extremen der Krümmung auf geschlossenen Kurven*, Festschrift, Heinrich Weber, 1912.

[2] W. Blaschke, *Die Minimalzahl der Scheitel einer geschlossenen konvexen Kurve*, Rendiconti del Circolo Matematico di Palermo, vol. 36 (1913), pp. 220–222; and *Vorlesungen über Differentialgeometrie*, Vol. I, p. 15.

plane, and $\rho(\phi)$ is the radius of curvature of the oval. This equation can be interpreted as saying that the center of gravity of a circle, loaded with mass of density, $\rho$, is at the center of the circle. Then, assuming that $\rho$ is continuous and has the same sign at all points on the curve, Blaschke shows that this is impossible if $\rho$ has only two extrema, and furthermore he shows that if the number of extrema of $\rho$ is finite, then they can occur only in an even number. H. Mohrmann[3] has also given a proof of this theorem.

G. Szegö[4] and W. Süss[5] have given proofs of a similar theorem which states that on any oval with a continuous radius of curvature, there are at least three pairs of points such that the tangents at the points of a pair are parallel, and the curvatures are there equal. Szegö proves it by the use of Fourier series, while Süss puts equation (1) in the form

$$\int_0^{\pi/2} \left[ s(\phi) \cos \phi - s\left(\frac{\pi}{2} + \phi\right) \sin \phi \right] d\phi = 0,$$

where $s(\phi) = \rho(\phi) - \rho(\pi + \phi)$, and then makes use of the continuity of $s(\phi)$ to show that $s(\phi)$ has at least three zeros in the interval, $0 \leqq \phi \leqq \pi$.

Of a similar nature is the theorem[6] in affine geometry that on every oval there are at least six sextactic points, a sextactic point being one where the osculating conic to the oval is stationary. We also have the theorem that there are six points on an oval where the affine normals pass through the center of gravity of the area enclosed by the oval.

Also of interest is the theorem of A. Emch[7] that in every oval at least one square can be inscribed. This is proved by the use of the lemma that two distinct rhombs with corresponding parallel sides or parallel axes can never be inscribed in the same oval. This theorem has been generalized by S. Kakeya[8] to read: In every oval there can be inscribed at least two rectangles similar to a given rectangle, except if the given rectangle is a square.

One of the most striking relations between geometry "im Grossen" and geometry "im Kleinen" is the theorem that if an oval is elliptically curved throughout (i.e., the osculating conic section at every point is an ellipse), any five points of the oval always lie on an ellipse.

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[3] H. Mohrmann, *Die Minimalzahl der Scheitel einer geschlossenen konvexen Kurve*, Rendicont del Circolo Matematico di Palermo, vol. 37 (1914), p. 267.

[4] G. Szegö, *Lösungen zu Aufgaben*, Archiv der Mathematik und Physik, vol. 28 (1920), p. 183.

[5] W. Süss, *Kurzer Beweis eines Satzes von W. Blaschke über Eilinien*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 33 (1924), 2 Abt. p. 32, or Tôhoku Mathematical Journal, vol. 24 (1925), pp. 66–67.

[6] Blaschke, *Differentialgeometrie*, vol. II, pp. 43–46.

[7] A. Emch, *Some properties of closed convex curves in a plane*, American Journal of Mathematics vol. 35 (1913), pp. 407–412.

[8] S. Kakeya, *On the inscribed rectangles of a closed convex curve*, Tôhoku Mathematical Journal vol. 9 (1916), pp. 163–166.

3. *The curvature-centroid and curvature-axis.* In 1838, J. Steiner[9] introduced the curvature-centroid as the point with respect to which the pedal curve of an oval had a minimum area. He showed geometrically that if the oval be considered as loaded with a mass density proportional to the curvature, its center of gravity is the curvature-centroid. N. M. Ferrers[10] and T. Hayashi[11] have given analytical proofs of this theorem. One of these depends on the following fact. If $(a, b)$ are the coordinates of the point under consideration, the area of the pedal curve is

$$A = \tfrac{1}{2} \oint [(x - a) \sin \phi - (y - b) \cos \phi]^2 d\phi.$$

It is easily shown that the values of $a$ and $b$ determined by the equations $\partial A/\partial a = \partial A/\partial b = 0$ are the coordinates of the center of gravity for the curve when loaded as above. Hayashi also showed that the curvature-centroids of an oval and its evolute are identical. By a method of proof similar to that outlined above for the four-vertex theorem, he showed that the radii vectors of an oval and its pedal curve have each at least four extrema.

B. Su[12] introduced a generalisation of the curvature-centroid by determining that line, which he calls the curvature-axis, about which the oval has a minimum moment of inertia, if it be loaded with a mass of density proportional to its curvature. He shows that the curvature axes of a parallel series of ovals form a pencil of lines, through the common curvature-centroid of the series, which reduce to a single line when and only when

$$\oint \rho(\phi) \sin 2\phi d\phi = \oint \rho(\phi) \cos 2\phi d\phi = 0.$$

In particular the curvature-axes of a series of parallel curves of constant breadth are one and the same line, since the curves of constant breadth satisfy the above conditions.

4. *The extremal chords of an oval.* T. Hayashi[13] gave the first solution of this problem. He divided it into two parts, first, the maximum chord in a given direction, and second, the extremals of these maximum chords. Then by very simple analytical considerations he was able to show that the tangents to the

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[9] J. Steiner, *Von dem Krümmungs-Schwerpuncte ebener Curven*, Crelle's Journal, vol. 21 (1838), pp. 33–63 and 101–133, or *Werke*, vol. 2, pp. 97–159.

[10] N. M. Ferrers, *Note on a geometrical theorem of Mr. Steiner*, Quarterly Journal of Pure and Applied Mathematics, vol. 4 (1861), pp. 92–94.

[11] T. Hayashi, *On Steiner's curvature-centroid*, Science Reports of the Tôhoku Imperial University, vol. 13 (1924), p. 109.

[12] B. Su, *On the curvature-axis of the convex closed curve*, Science Reports of the Tôhoku Imperial University, vol. 17 (1928), pp. 35–42.

[13] T. Hayashi, *The extremal chords of an oval*, Tôhoku Mathematical Journal, vol. 22 (1923), pp. 387–393.

oval at the extremities of the maximum chord in a given direction are parallel, and that the tangents at the extremities of an extremal chord are perpendicular to the chord, i.e., the extremal chords are double normals of the oval.

5. *Various Inequalities.* The majority of the theorems on ovals are relations giving the upper or lower limits to ratios between the area and perimeter, the area and breadth, etc. Fundamental among these is the following inequality,[14] valid for the affine as well as for the Euclidean plane:

$$2/F_{22}\sqrt{(F_{12}^2 - F_{11}F_{22})} \geqq \max\sqrt{(D_1/D_2)} - \min\sqrt{(D_1/D_2)},$$

where $F_{11}$, $F_{22}$ are the areas of two ovals $K_1$ and $K_2$ and $D_1$, $D_2$ are the areas of similar and similarly oriented triangles circumscribed about the ovals $K_1$ and $K_2$, respectively. $F_{12}$ is the Minkowski mixed area of the two ovals defined by the equation,

$$F_{12} = \mathfrak{S}(x_1 dy_2 - y_1 dx_2).$$

If we let $K_2$ be a unit circle, we get as a special case of this inequality the theorem, valid only in the Euclidean plane, that

$$L^2 - 4\pi F \geqq \pi^2(\max r - \min r)^2, \tag{2}$$

where $L$, $F$ are the perimeter and area of an oval and $r$ is the radius of a circle inscribed in a triangle which is circumscribed about the oval.

This is a sharper relation than $L^2 - 4\pi F \geqq 0$ which expresses the well-known isoperimetric property of the circle, a property recognized by the Greek geometers, but first rigorously proved by Minkowski.[15]

If we represent by $\Delta$ the breadth, or minimum separation of two parallel tangents, and by $D$ the diameter, or maximum separation of two parallel tangents, of an oval, we have, among others, the following expressions for the lower limit to the area of an oval:

$$\frac{L^2 - 4D^2}{12};\quad \frac{(L-2D)\sqrt{(4LD - L^2)}}{4};\quad \frac{D(L-2D)}{3}.$$

The first two were proved by T. Kubota[16] from the following two lemmas:

I. If we are given any polygon $B_1B_2 \cdots B_k$ of perimeter $L$ and such that $B_1B_2 = B_1B_3 = \cdots = B_1B_k = D$, otherwise $B_iB_{i+1} < D$ then it is possible to construct a quadrilateral $C_1C_2C_3C_4$ of perimeter $L$ such that

$$C_1C_2 = C_1C_3 = C_2C_3 = C_1C_4 = D \text{ and hence } C_3C_4 = L - 3D$$

with an area less than that of $B_1B_2 \cdots B_k$.

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[14] W. Blaschke, *Eine Verschärfung von Minkowskis Ungleichheit für den gemischten Flächeninhalt*, Hamburger Abhandlungen, vol. 1 (1922), pp. 206–209.

[15] H. Minkowski, *Gesammelte Abhandlungen*, vol. 2, pp. 242–249.

[16] T. Kubota, *Eine Ungleichheitsbeziehung über die Eilinien*, Tôhoku Mathematical Journal, vol. 24 (1925), pp. 60–63. T. Kubota, *Einige Ungleichheitsbeziehungen über Eilinien und Eiflächen*, Science Reports of the Tôhoku Imperial University, vol. 12 (1923), pp. 45–65.

II. If we have a convex polygon $A_1A_2 \cdots A_n$ with fixed values of $D$ and $L$, then we can construct a polygon $B_1B_2 \cdots B_k$, of area equal to or less than that of $A_1A_2 \cdots A_n$, and having the properties assummed in lemma I.

The area reaches the lower limit given by the first expression only when the oval consists of a straight line counted doubly.

The third relation can be derived from the first by making use of the fact that $L \geqq 2D$.

For upper limits to the area of an oval we have the following, in addition to (2) above:

$$\tfrac{1}{4}\pi D^2;\ \tfrac{1}{4}(2\Delta L - \pi\Delta^2).$$

These were also established by Kubota.[17]

Rosenthal and Szász have shown[18] that $L \leqq \pi D$, where the equality holds only for curves of constant breadth.

If we let $p(\phi) = -x\sin\phi + y\cos\phi$ be the Minkowski "Stützgerade Funktion" we have

$$L = \oint \rho d\phi \text{ and } F = \tfrac{1}{2}\oint p\rho d\phi$$

from which can easily be derived the theorem,[19] due to Hurwitz and Blaschke, that the perimeter or area of an oval lies between the perimeters or areas, respectively, of the largest and smallest osculating circles.

From the same expressions we also find that

$$R \geqq \tfrac{1}{2}D(1 - 2DL^{-1}) \text{ and } r \leqq \tfrac{1}{2}D,$$

where $R$, $r$ are the radii of the largest and smallest osculating circles of the oval.

By making use of a Fourier expansion of $\rho$ and $1/\rho$ it can be shown that the average value of *the radius of curvature* of an oval, averaged with respect to $s$, is equal to or greater than the *radius* of a circle with the same perimeter; and the *curvature* of an oval, averaged with respect to $\phi$, is equal to or greater than the *curvature* of a circle with the same perimeter.

In the affine plane we have the two following relations:[20]

$$8\pi^2F - S^3 \geqq 0;\ 4\pi A - (3\sqrt{3})F \geqq 0,$$

where $F$ is the area, $S$ the affine perimeter, and $A$ the area of the largest inscribed triangle, of an oval.

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[17] T. Kubota, *Einige Ungleichheitsbeziehungen über Eilinien und Eiflächen*, Science Reports of the Tôhoku Imperial University, vol. 12 (1923), pp. 45–65.

[18] Rosenthal and Szász, *Eine Extremaleigenschaft der Kurven konstanter Breite*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 25 (1917), pp. 278–282.

[19] T. Hayashi, *The extremal chords of an oval*, Tôhoku Mathematical Journal, vol. 22 (1923), pp. 387–393.

[20] W. Blaschke, *Differentialgeometrie*, vol. 2, pp. 50, 61.

We also have the theorem[21] that for an oval which is elliptically curved throughout, the area or affine perimeter of the largest osculating ellipse is equal to or greater than the area or affine perimeter, respectively, of the oval; and the equality holds only when the oval is an ellipse. This corresponds to one half of the theorem concerning osculating circles mentioned above.

6. *Conclusion.* There are still a large number of unsolved problems in this field. For instance Blaschke has proposed the following theorems:

I. If a point lies inside all osculating circles of an oval, it also lies inside every circle which has more than two points in common with the oval.

II. If a line lies outside all osculating circles of an oval, it does not touch any circle having more than two intersections with the oval.

Intuitively these theorems seem almost self-evident, but apparently no rigorous proofs have yet been given for them.

It would be very interesting to find out whether the affine analogon of the curvature-centroid has any interesting properties.

It would also be desirable to have a more geometrical discussion of the extremal chords of an oval.

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## ON THE EXPRESSION OF AN INTEGER AS THE SUM OF AN ARITHMETIC SERIES

By LAURENS EARLE BUSH, University of North Carolina

T. E. Mason in the American Mathematical Monthly, vol. 19 (1912), p. 46, found the number of ways in which an integer can be expressed as the sum of consecutive integers or of consecutive positive integers. In this paper we shall consider the more general question of expressing an integer as the sum of an arithmetic series of integers or of positive integers.

Without loss of generality we may consider the expression of positive integers only, since corresponding theorems for negative integers follow immediately from those for positive integers. We shall consider an integer itself as an arithmetic series of one term, and such a series we shall call *trivial*. In the theorems which follow trivial series are counted unless specifically excluded. We also admit zero as an integer and consider a series beginning or ending with zero as different from the same series with the zero suppressed.

Let $K = 2^{e_0}p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$ be any positive integer, where $p_1, p_2, \cdots, p_r$ are distinct odd primes:

Theorem 1. *The number of different ways in which K can be expressed as the sum of an arithmetic series of integers with a specified even common difference, d,*

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[21] T. Kubota, *Eine Bemerkung zur affinen Geometrie*, Science Reports of the Tôhoku Imperial University, vol. 12 (1923), pp. 1–5.

*is the same as the number of distinct positive divisors of $K$, i.e., in $\Pi_{i=0}^{r}(e_i+1)$ ways.*

For, if $K$ is the sum of such a series beginning with the integer $a$ and having $n$ terms, then

$$K = \tfrac{1}{2}n[2a + (n-1)d],$$

which, solved for $a$, gives

$$a = \frac{K}{n} - (n-1)\frac{d}{2}. \tag{1}$$

Since $d$ is even, $n$ must be a divisor of $K$. Conversely, every distinct positive divisor, $n$, of $K$ leads to a series of the required type containing $n$ terms.

Corollary: *Every composite integer can be expressed as the sum of a non-trivial arithmetic series of integers with a specified even common difference, d, in at least two different ways; every prime in one, and only one, way; and unity can not be so expressed.*

Theorem 2: *The number of different ways in which $K$ can be expressed as the sum of an arithmetic series of integers with a specified odd common difference, $d$, is twice the number of distinct positive odd divisors of $K$, i.e., in $2\Pi_{i=}^{r}(e_i+1)$ ways.*

For, if $K$ is the sum of such a series beginning with the integer $a$ and having $n$ terms, $a$ has the value given by (1). We have two cases, according as $n$ is odd or even.

If $n$ is odd, the last term in (1) is an integer and $n$ must be a divisor of $K$. Conversely, every positive odd divisor, $\delta$, of $K$ leads to a series of the required type containing $\delta$ terms.

If $n$ is even, the last term in (1) is half of an odd integer, therefore $K/n$ must also be half of an odd integer, or $\delta = 2K/n = K/\frac{1}{2}n$ must be an odd divisor of $K$. Conversely, every positive odd divisor, $\delta$, of $K$ leads to a series of the required type containing $2K/\delta$ terms.

Hence, each distinct positive odd divisor of $K$ leads to two different expressions of $K$ of the required type, one having an odd number of terms, the other an even number. This leads to the

Corollary: *Exactly half of the different expressions of $K$ as the sum of an arithmetic series of integers with a specified odd common difference, d, have an even number of terms.*

If $d=1$ we have one of the results obtained by Mason:

Corollary: *The number of different ways in which $K$ can be expressed as the sum of consecutive integers is $2\Pi_{i=1}^{r}(e_i+1)$.*

We may also state the

Corollary: *Every composite integer can be expressed as the sum of a non-trivial arithmetic series of integers with a specified odd common difference, d, in at least one way; a power of 2 in only one way; an odd prime in three, and only three, different ways; and unity in one, and only one, way.*

It is to be noted that in the above corollary and in the corollary to theorem 1, if $d = \pm K$, we are admitting as non-trivial the series of two terms consisting of zero and $K$. For example, unity can be expressed as the sum of two or more consecutive integers only in this way, but, for $d = 3$, we have $1 = -1 + 2$.

In the theorems which follow we shall for convenience in notation consider only positive and zero common differences. That in doing this we lose no generality is evident, since any expression for $K$ as the sum of a series with common difference $d$ becomes a similar expression with common difference $-d$ if written in reverse order. In fact, the theorems stated below are equally true for negative values of $d$ if we replace $d$ by its absolute value in (2) and (4).

Theorem 3: *The number of different ways in which $K$ can be expressed as the sum of an arithmetic series of positive integers with a specified positive even or zero common difference, $d$, is the number of distinct positive divisors, $\delta$, of $K$ which satisfy the inequality.*

$$\delta(\delta - 1)d < 2K. \tag{2}$$

For, if the series is to have only positive terms it is both necessary and sufficient that the smallest term be positive. Since $d$ is positive the first term is the smallest, and from (1) we obtain the condition

$$n(n - 1)d < 2K. \tag{3}$$

Since the permissible values of $n$ are those, and only those, which are divisors of $K$, the theorem follows.

Corollary: *A prime number can not be expressed as the sum of a non-trivial arithmetic series of positive integers with a positive even common difference, $d$.*

For, if $K$ is prime, its only positive divisors are unity and itself. Unity leads to the trivial expression, while $K$ does not satisfy (2), for, since $d \geqq 2$ and $K \geqq 2$,

$$K(K - 1)d \geqq 2K(K - 1) \geqq 2K.$$

Theorem 4. *The number of different ways in which $K$ can be expressed as the sum of an arithmetic series of positive integers with a specified positive odd common difference, $d$, is equal to the number of distinct positive odd divisors, $\delta$, of $K$ which satisfy inequality* (2) *plus the number of distinct positive odd divisors, $\delta$, of $K$ which satisfy the inequality*

$$\delta(\delta + d) > 2Kd. \tag{4}$$

For, as in the last theorem, (3) is a necessary and sufficient condition that the series contain only positive terms. However, when $d$ is odd each odd divisor, $\delta$, leads to two values of $n$, namely, $n = \delta$ and $n = 2K/\delta$. Hence, those, and only those, values $\delta$ which satisfy (2) will lead to a series of an odd number of positive terms; while those, and only those, which satisfy

$$\frac{2K}{\delta}\left(\frac{2K}{\delta} - 1\right)d < 2K,$$

or its equivalent (4), will lead to a series of an even number of positive terms.

Corollary: *A power of two is not expressible as the sum of a non-trivial arithmetic series of positive integers with a positive odd common difference.*

For a power of two has only the odd positive divisor unity, which does not satisfy (4), since $K \geqq 1$ and $d \geqq 1$, and therefore

$$\delta(\delta+d) = 1+d \leqq (2K-1)\ d+d = 2Kd.$$

Hence, the trivial series is the only one.

Corollary: *An odd prime, $K$, can be expressed as the sum of a non-trivial arithmetic series of positive integers with a specified positive odd common difference, $d$, such that $d<K$, in one, and only one, way.*

For, the only odd divisors of $K$ are unity and $K$. Unity does not satisfy (4) as shown above. $K$ does not satisfy (2), for $K \geqq 3$, $d \geqq 1$, and $\delta(\delta-1)d = K(K-1)d \geqq 2K$. But the prime $K$ always satisfies (4), for since $d<K$,

$$K(K+d) = K^2 + Kd > Kd + Kd = 2Kd.$$

Therefore, there is one, and only one, expression for $K$ other than the trivial one.

Another of Mason's results is contained in the following

Corollary: *The number of ways in which $K$ can be expressed as the sum of consecutive positive integers is equal to the number of positive odd divisors of $K$, i.e., in $\Pi_{i=1}^{r}(e_i+1)$ ways.*

For, we have $d=1$ and (2) and (4) reduce to $\delta(\delta-1)<2K$ and $\delta(\delta+1)>2K$. Any odd divisor, $\delta$, of $K$ which does not satisfy one of these relations will evidently satisfy the other. Furthermore, the same positive odd divisor, $\delta$, can not satisfy both, for if it did, we should have

$$\delta^2 < 2K + \delta \quad \text{and} \quad \delta^2 > 2K - \delta,$$

i. e.

$$\delta < 2\frac{K}{\delta} + 1 \quad \text{and} \quad \delta > 2\frac{K}{\delta} - 1.$$

Since $K/\delta$ is an integer, we have an odd integer, $\delta$, lying between the two consecutive odd integers $2(K/\delta)-1$ and $2(K/\delta)+1$, which is impossible.

Corollary: *A necessary and sufficient condition that $K$ be expressible as the sum of two or more consecutive positive integers is that it be not a power of 2.*

In order to illustrate the theorems stated above, let us take $K = 12 = 2^2 \cdot 3$. The divisors of 12 are six in number, namely, 1, 2, 3, 4, 6, and 12, of which two, namely, 1 and 3, are odd. If $d=2$, by theorem 1, we have six expressions for 12:

$$12 = 12$$

$$12 = 5 + 7,$$

$$12 = 2 + 4 + 6,$$

$$12 = 0 + 2 + 4 + 6,$$
$$12 = -3 - 1 + 1 + 3 + 5 + 7,$$
$$12 = -10 - 8 - 6 - 4 - 2 + 0 + 2 + 4 + 6 + 8 + 10 + 12.$$

By theorem 3, since for $d=2$, (2) is satisfied by the divisors 1, 2, and 3 only, we have the first three of these expressions alone consisting of positive integers. If $d=3$, by theorem 2, we have four expressions for 12:

$$12 = 12,$$
$$12 = 1 + 4 + 7,$$
$$12 = -34 - 31 - 28 - 25 - 22 - 19 - 16 - 13 - 10 - 7 - 4 - 1 + 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32 + 35,$$
$$12 = -9 - 6 - 3 + 0 + 3 + 6 + 9 + 12.$$

By theorem 4, since for $d=3$, (2) is satisfied by the odd divisors 1 and 3 only, while no odd divisor satisfies (4), we have the first two of these expressions alone consisting of positive integers. The only expressions of 12 as the sum of consecutive integers are

$$12 = 12,$$
$$12 = 3 + 4 + 5,$$
$$12 = -11 - 10 - 9 - 8 - 7 - 6 - 5 - 4 - 3 - 2 - 1 + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12,$$
$$12 = -2 - 1 + 0 + 1 + 2 + 3 + 4 + 5.$$

Of these, only the first two consist of positive integers.

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## A SHORT ACCOUNT OF THE HISTORY OF SYMMETRIC FUNCTIONS OF ROOTS OF EQUATIONS

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A function is symmetric with respect to any number of letters if an interchange of any two of the letters does not change the function. Thus $\alpha\beta\gamma$ and $\alpha+\beta+\gamma+\alpha^2\beta^2\gamma^2$ are symmetric functions. The simplest ones are those involved in the relations between the coefficients and roots of a quadratic equation. If $\alpha$ and $\beta$ are the roots of the equation $x^2+bx+c=0$, then $\alpha+\beta=-b$ and $\alpha\beta=c$, $b$ and $c$ being symmetric functions of the roots $\alpha$ and $\beta$.

It shall be the purpose of this paper to seek out the first evidence of the knowledge of the relationship between the coefficients and roots of an equation;

to discover, if possible, the faint beginnings or shadowy forerunners of the subject of symmetric functions, to trace their historical development up to the nineteenth century, and to discuss briefly the mathematicians who contributed to the progress of the theory.

The study is not exhaustive and the aim has been to present in concise form an account of the early factors which have made this phase of algebra what it is today, and to supply the student of the theory of equations with some historical material in this specific part of his subject. It has been my particular experience to find that new interest and vitality come to a subject when there is linked up with it a knowledge of its origin and the vicissitudes of its progress, and an acquaintance with those renowned and worthy scholars of previous generations who have provided such a noble heritage for the student of today. One approaches his work with a somewhat different conception of it when he realizes that he is sampling a stream that has been flowing steadily for centuries and will continue to flow in like manner after he is gone.

Original sources in the early history of the subject have naturally been difficult to find and interpret. In preparing this paper I have been fortunate in having at my disposal the library of Professor David Eugene Smith of Teachers College, Columbia University, in addition to the library of the American Mathematical Society and the New York Public Library. These libraries, especially the first named, have provided such a wealth of material as to make the study profitable and interesting.

Original sources have been consulted in all instances mentioned with the exception of Peletier, Vieta and Vandermonde. The authority for the statements in regard to the first two men is Charles Hutton—*Mathematical and Philosophical Dictionary* (London, 1796). For a reproduction of Vieta's work on roots and coefficients, I have consulted Francis Maseres—*Tracts on the Resolution of Cubic and Biquadratick Equations* (London, 1803), page 531. The basis for the paragraph on Vandermonde is a statement on page 51 of Faà de Bruno's *Théorie des Formes Binaires* (1876).

As is the case with the development of algebra as a whole, the subject of symmetric functions to a perhaps greater extent waited upon the introduction and improvement of symbolism. It was hard to perceive with any clarity relations between roots and coefficients when the equation was wrapped up in a paragraph of words representing it. Prior to the time of Franciscus Vieta (1540–1603), who was among the first to employ letters to represent numbers, we find very little trace of the thing we are seeking. We might say that "algebraic shorthand" was almost essential to the development of the subject. It was only when the eye was able to aid the mind that progress began to be noticeable.

Luca Pacioli (c.1445–1509) in his *Sūma* (1494), a general summary of mathematics up to that time, devotes a part to algebra under the name—L'Arte Magiore; ditta dal vulgo la Regola de la Cosa, over Alghebra e Amucabala.[1] In

[1] *Sūma de Arithmetica, Geometria, Proportioni e Proportionalita*, Venice, 1494, folio 67.

this he solves equations by completing the square and seems generally to have a clear notion of the quadratic. Although I can find no mention of it, he very probably recognized some relation between the roots and the numerical parts of the equation.

The starting point of our subject seems to be with that versatile and eccentric Italian mathematician of the middle of the sixteenth century, Girolamo Cardano (1501–1576), whose name has come down in connection with the first solution of the cubic equation. His work on algebra, *Ars Magna*, was published in 1545. In this he gives an equation,[2] $x^3+72=11x^2$ and the roots $\sqrt{40}-4$, $\sqrt{40}+4$, 3. It should be noticed that the correct roots are $-\sqrt{40}+4$, $\sqrt{40}+4$, 3; but this fault does not affect his knowledge of the relations involved. He says that the difference between the positive and the negative roots is always equal to the coefficient of the second term—"differentia aequationum verarum et fictarū semper est numerus quadratorū." He adds $\sqrt{40}+4$ and 3. From this sum he subtracts $\sqrt{40}-4$ obtaining 11, the coefficient of $x^2$.

He also shows, in effect, that when the second term is missing the sum of the negative roots equals the sum of the positive ones.

Jacques Peletier (1517–1582), writing on algebra thirteen years later, gives a method of finding the roots of an equation among the divisors of the absolute term when the root is rational, whether integral or fractional, and he observes that the root always "lies hid" in the term and is some one of its divisors.[3]

Rafael Bombelli (born c. 1530) in his algebra[4] published in 1572 shows about the same knowledge as that set forth by Cardan.

Vieta's work on algebra, which gave a great impetus to all branches of algebra and especially to that which we are considering, did not become known until after his death. It was published by Alexander Anderson in 1615. Chapter 14 of this work contains in four theorems the general relation between the roots of an equation and the coefficients of its terms when all the roots are positive. The first theorem in Vieta's original style is as follows:

Si B+D in A−A quadratum aequatur
B in D, A explicabilis est de qualibet illarum duarum B vel D.
3N−1Q aequetur 2. Fit 1N 1 vel 2.

This in modern notation is

If $(a+b)x-x^2=ab$; then $x=a$ or $b$.
$3x-x^2=2$, $x=1$ or 2.

The cubic, biquadratic and quintic are given in similar fashion.

This seems to show that Vieta had a very clear idea of the relation he expresses but such is not quite the case. I quote from Cajori's *History of Elementary Mathematics:*

"Vieta arrived at a partial knowledge of the relation existing between the coefficients and the roots of an equation. Unfortunately he rejected all except

[2] *Artis Magnae, sive de regulis algebraicis*, 2nd edition, Basel, 1570, p. 10.
[3] *L'algèbre départie en deux livres*, Lyons, 1554.
[4] *L'algebra parte maggiore dell' arimetica divisa in tre libri*, Bologna, 1572.

positive roots and could not therefore fully perceive the relations in question. His nearest approach to complete recognition of the facts is contained in the statement that the equation $x^3-(u+v+w)x^2+(uv+vw+uw)x-uvw=0$ has three roots $u, v, w$. For cubics this statement is perfect if $u, v, w$, are allowed to represent any numbers. But Vieta is in the habit of assigning to letters only positive values so that the passage means less than at first sight it appears to do."[5]

However we can appreciate how great an advance Vieta made when there is considered the few fragmentary statements that have come from his predecessors. He paves the way and provides an introduction to the first man who really has a place in the history of symmetric functions of roots of equations, a man, who for clearness and grasp of material at hand in not only this topic but also in other phases of algebra could well hold his place a century later.

The man was Albert Girard (1595–1632) and his work on algebra is a little 34-leaf pamphlet called *Invention Nouvelle en l'Algèbre*, published in 1629.

Girard gives the triangle later known as Pascal's triangle and uses it as the basis for developing a theorem on symmetric functions, although he had no idea of them as such. It is interesting to note how he makes and interprets the triangle. I shall translate his words as literally as possible:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

"When many units are put as on the side and other numbers in the middle, you find such a figure by means of addition. It may be called the triangle of extraction. The unity above signifies simple arithmetic and the others, algebra, i.e. 1, 1 is the degree of (1) and 1,2,1 the degree of (2), then 1,3,3,1 is called the degree of (3) and thus always to infinity."[6]

The symbol (1) is a substitute for Girard's* notation for the first power of the unknown; (2) and (3) are respectively symbols for the second and third powers.

Designating by "première faction" the sum of a group of numbers; "deuxième faction" the sum taken two at a time, etc., he states the theorem—"If a multitude of numbers is proposed, the multitude of the products of each faction can be expressed by the triangle of extraction and by the same rank as the multitude of numbers." Thus, if there are four numbers, the row 1,4,6,4,1 signifying the fourth power, is taken. The 1 denotes the unit of the greatest, the 4 the first faction, the 6 the second faction, etc.

In an illustrative equation, $1\,(4)=4\,(3)+7\,(2)-34\,(1)+24$, he states that the exponent of the highest quantity is 4 which signifies that there are four solutions, neither more nor less. But he says "in order to see the thing in its

[5] Revised edition, New York, 1917, page 230.

[6] Loc. cit., page 42.

* Girard's symbol was a circle enclosing the number. Parentheses instead of circles are used here for convenience in printing.

perfection" it is necessary to take the signs arranged in a different order, thus

$$1\ (4)-7\ (2)-24\ (0)=4\ (3)-34\ (1);$$

then the coefficients with their proper signs are 4, $-7$, $-34$, $-24$, which are the four factions of the four solutions.[7]

He gives an equation, $x^4-4x+3=0$, which has two imaginary roots and shows that the rule for forming the coefficients from the roots holds true in all cases.

Girard arrives rather casually and uniquely at the development of the sums of the powers of the roots in terms of the coefficients. He is discussing the above mentioned relationship between coefficients and factions when he assumes that someone for the sake of simplicity, instead of saying the sum, the product two by two, three by three, etc., may like to say the sum, the sum of the squares, the sum of the cubes, etc. In order to show that these two ideas do not represent the same thing, he gives the following example:

"If the coefficients of the second, third, fourth terms, etc. are $A$, $B$, $C$, etc., then in any degree of equation,

| | | |
|---|---|---|
| $A$ | will | solutions |
| $A^2-2B$ | be | squares |
| | the | |
| $A^3-3AB+3C$ | sum | cubes |
| $A^4-4A^2B+4AC+2B^2-4D$ | of | biquadrates"[8] |

This is the well known theorem which bears the name of Newton, but we find it given almost as completely a century before by Girard.

Charles Hutton (1737–1823), in his *Mathematical and Philosophical Dictionary* (1795–6), puts his stamp upon Girard in these words:

"He was the first person who understood the general doctrine of the formation of the coefficients of the powers from the sum of the roots and their products. He was the first who discovered the rules for summing the powers of the roots of any equation."[9]

Thomas Harriot (1560–1621) at about the same time as Girard was also discovering relations of the roots to coefficients, but he lacked the insight and knowledge of his contemporary. His work on algebra was published ten years after his death by his friend, Walter Warner. It was probably written[10] about 1610. He studied equations from the standpoint of their formation from binomial factors or roots and thus recognized the relation between the roots and coefficients when the roots are positive. The following is an example of his method:

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[7] Loc. cit., p. 43.
[8] Loc. cit., p. 46.
[9] Vol. 1, p. 63.
[10] D. E. Smith, *History of Mathematics* (1923), Vol. 1, p. 388.

$$\begin{array}{ll} a=+b & a-b=0 \\ a=+c & a-c=0 \\ a=+d & a-d=0 \end{array}$$

$$aaa-baa+bca$$
$$-caa+bda$$
$$-daa+cda-bcd=0$$[11]

Cajori says, "He was the first to decompose equations into their simple factors, but he failed to recognize imaginary or even negative roots."[12]

William Oughtred (1574–1660) in the *Clavis Mathematicae* published in 1631 follows Vieta to a large extent but does not extend the relation we are studying.

For the next seventy-five years, up to the time of Newton, there is little worth noting in the development of symmetric functions. René Descartes' geometry, which first appeared in 1637, begins with a discussion of the nature and roots of equations. Descartes generates equations in a manner similar to Harriot by the multiplication of binomial factors. He shows how an equation can be depressed by a binomial composed of the unknown plus or minus a root. Being interested in equations only from the geometrical viewpoint, he proceeds to their transformation with no further regard for roots and coefficients as such.

In 1673 appeared the unique work of John Wallis (1616–1703), *De Tractatus Algebra, Historicus et Practicus*. Wallis devotes a short chapter to the subject, "The composition of coefficients." This relates to the manifest number of roots that any equation can have and to the ease of perceiving that "the coefficient of the second term, reckoning downward from the highest, is the aggregate of all the roots, the coefficient of the third term is the aggregate of the rectangles," etc.[13] This is the sum total of his contribution and he either did not know of Girard's work or considered it of not sufficient importance to mention. The former seems more likely. Also, this pioneer historian was so wrapped up in Harriot's algebra that he had eyes for little else.

As is often the case, the man who makes a rediscovery of some important truth, or gives his results to the world under more favorable circumstances than the original discoverer, gathers to himself the honor of having his name perpetuated while that of the former goes unsung. Such is the fact that we note in this connection with reference to Girard and Newton. Almost a hundred years after the time of Girard the *Arithmetica Universalis* of Isaac Newton (1642–1727) appeared, and in it, under the heading of transmutation of equations, is the theorem on the sums of powers of the roots. From this time on, every work on symmetric functions begins with a presentation of "Newton's theorem."

For the sake of comparison it may be worth while to give it as Newton did in 1707:

[11] *Artis Analyticae Praxis ad Aequationes Algebraicas .... Resoluendas*, London, 1631, p. 18.
[12] *History of Mathematics* (1919), page 157.
[13] English edition, London, 1685, p. 142.

"Let us suppose now, that the known quantities of the terms of any equation under the signs changed are $p, q, r, s, t, v$, viz., that of the second $p$, that of the third $q$, of the fourth $r$, of the fifth $s$, and so on. And the signs of the terms being rightly observed, make $p=a$, $pa+2q=b$, $pb+qa+3r=c$, $pc+qb+ra+4s=d$, $pd+qc+rb+sa+5t=e$, $pe+qd+rc+sb+ta+6v=f$, and so ad infinitum, observing the series of the progressions.

"And $a$ will be the sum of the roots, $b$ the sum of the squares of each of the roots, $c$ the sum of the cubes, $d$ the sum of the biquadrates, $e$ the sum of the quadrato-cubes, $f$ the sum of the cubo-cubes, and so on."[14]

Newton gives no proof or demonstration of how he obtained this theorem, but passes on immediately to the use of the sums of powers in determining the limits of the roots of equations. In an English edition of *Arithmetica Universalis* in 1769, a commentary by Rev. Theaker Wilder shows the derivation of the theorem, stating that it "follows easily from the algebraical expression of the quantities and the binomial theorem."[15]

The middle of the eighteenth century finds the relation between roots and coefficients fairly well understood. Crousaz's algebra in 1726, Saunderson's in 1740, Clairaut's in 1749, and Maclaurin's in 1748, all contain chapters on roots and coefficients.

In the years just following Newton a number of mathematicians were interested in demonstrating and proving the Newtonian formula. Among the most important proofs may be mentioned Maclaurin's[16] in 1748 and Euler's[17] in 1750. The latter is called a double demonstration of Newton's theorem, the first of which is by the use of logarithmic differentiation and development in series, the second being algebraic and elementary in type. The second, however, is an excellent proof, the demonstration of which has been characterized by Leopold Kronecker (1823–1891) as very exact and elegant. It is similar to that of Maclaurin's in finding the sum of the $r$th powers in an equation of the $n$th degree when $r \geqq n$, but differs for the case of $r < n$.

Jean Castillon (1709–1791), who edited an edition of the *Arithmetica Universalis* in 1761, includes a demonstration in an extensive commentary on the theorem.[18] In an appendix to the above edition there is also a proof by Georg F. Baermanns (1717–1769). He writes that he early found out the usefulness of the theorem in the theory of limits of equations and that he deplored the lack of a proper demonstration among the mathematicians of his time. After searching in vain he at last himself developed a "praiseworthy" one. He reverses

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[14] *Arithmetica universalis sive de compositione et resolutione arithmetica liber*, Cambridge, 1707, p. 251.

[15] Page 393.

[16] *A Treatise of Algebra*, London, 1748, pp. 286 ff.

[17] *Gemina theormatis Neutonioni, quo traditur relatio inter coefficientes cuius vis aequationis algebraicae et summas potestatum radicum eiusdem*, Opuscula Varii Argumenti, 1750, vol. 2, p. 108.

[18] Page 70.

common practice in representing coefficients by Greek letters and he makes use of increments and a sort of differential notation.

As we scan the progress of the subject after the impress of Newton upon it, we find that the next man whose name is known to modern students of symmetric functions is Dr. Edward Waring, Lucasian Professor at Cambridge from 1760 until 1798. He was one of the first to write extensively on the theory of equations and he developed three important theorems on symmetric functions. These appear in Chapter I of his *Meditationes Algebraicae*, published in 1770.

The first theorem is a formula for finding immediately, as a function of the coefficients, the sum of the $n$th powers of the roots. Waring does not make known the method which leads to this theorem. The second theorem, the least useful of the three, appears as a formula for finding the sums of all the powers of the roots. The third contains a method for finding any rational and entire symmetric function of the roots in terms of the coefficients. This theorem is the beginning of the fundamental theorem of modern symmetric functions.

To Alexandre Théophile Vandermonde (1735–1796), chemist, musician, and political economist as well as mathematician, belongs the distinction of having first constructed tables of symmetric functions. Coming into possession of Waring's formulas a year after the publication of *Meditationes Algebraicae*, he worked out a set of these functions.[19] However, he gave no method for their calculation, and his work, being little known, was overshadowed by the results of Hirsch twenty-five years later.

At almost exactly the same time that Waring was working out his first theorem, Leonard Euler (1707–1783), in Russia, was directing his thoughts along the same lines with the result that he also obtained a like theorem. His conclusions were embodied in a memoir read before the Saint Petersburg Academy in January, 1771, under the title *Observationes Circa Radices Aequationum*, which was published the same year. One of the most important applications of the theory of symmetric functions, the method of elimination, is due to Euler.[20] This method which finds the final equation by means of symmetric functions was first presented by him before the Berlin Academy of Sciences in 1748. Two years later, Gabriel Cramer (1704–1752) in his treatment on algebraic curves[21] followed the same method in a somewhat more general way.

Among the great mathematicians of the eighteenth century who were interested in symmetric functions was Joseph Louis Lagrange (1736–1813) whose work, however, must be regarded not as an end but rather as a means to an end. He studied symmetric functions for the purpose of developing and improving methods for the solution of higher equations.

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[19] Mémoires de l'Académie des Sciences (1771).

[20] Cajori, *History of Mathematics*, page 235. Meyer Hirsch, *Literal Calculus and Algebra*, translated by J. A. Ross (1827), page 143.

[21] *Introduction à l'Analyses des Lignes Courbes* (Geneva, 1750), page 660.

Lagrange's first edition of the *Traité de la Résolution des Équations Numériques de tous les Degrés* appeared in 1798. In a note on the method of approximation by recurring series, he develops the algebraic division method of proving Newton's theorem.[22] This is perhaps the simplest of all methods and is one of several generally given in modern textbooks.

In this work of Lagrange's there is one of the first uses of the modern symbol $\sum$ for summation in connection with symmetric functions. The symbol appears before in Wallis's algebra but is used there for "cosine." Its introduction as the symbol for summation is due to Euler[23] in 1755. Lagrange is also the first to state and fully recognize the theorem that every rational and entire function of the roots can be expressed in terms of the coefficients. He says of this theorem "C'est la un des principes les plus féconds de la théorie des équations."[24]

Chief among Lagrange's contributions is a noteworthy formula[25] which Louis Poinsot (1777–1859) praised "aussi remarquable par son élégance que par sa généralité." It gives an expression for finding directly the sum of like negative powers of the roots of an equation. This theorem has been the foundation of a number of researches by Lagrange's successors, among whom the most important have been Cauchy, who has given a number of demonstrations of it,[26] and Eugène Rouche[27] who gave a rigorous and yet simple proof.[28] Lagrange also developed Newton's theorem by means of logarithms.

The first presentation of symmetric functions in a modern way is due to Meyer Hirsch (1765–1851). In the preface to his algebra (1809) he says, "I begin with symmetric functions; they are the foundation of all others."[29] Then follow several proofs of Newton's theorem and a general demonstration that every rational symmetric function of the roots can be expressed as a function of the coefficients. The first tables of symmetric functions that have become generally known and used were constructed by Hirsch. These contain all values up to equations of the tenth degree. He constructed these by means of the functions of the like powers of the roots, giving demonstrations of his method of work. This method requires excessive calculation when the functions are extended beyond the simpler forms, and hence was supplanted by the later formulas of Cayley and Sylvester.

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[22] *Traité de la Resolution des Équations Numériques* (Second edition, Paris, 1808), p. 133.

[23] Johannes Tropfke, *Geschichte der Elementar-Mathematik* (Berlin, 1921), Band II, page 33.

[24] Op. cit., p. 193.

[25] Memoirs of Berlin Academy, 1768, *Traité de la Résolution des Équations Numériques*, page 211.

[26] *Mémoire sur Quelques Séries Analogues à la Série de Lagrange*, Mémoires de l'Académie des Sciences (1830), Tome IX, page 104. *Oeuvres complètes d'Augustin Cauchy* (Paris 1882–1911), Séries I, Tome II, page 73.

[27] Journal de l'École Polytechnique, vol. 22, p. 193 ff.

[28] J. A. Serret, *Cours d'Algèbre Supérieure* (3rd edition, 1866), page 462.

[29] *Sammlung von Aufgaben aus de Theorie der Algebraischen Gleichungen*, Berlin, 1809, page ix.

# BOND YIELDS AND BOND PRICES

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1. *Introduction.* Let $P$ be the purchase price of a bond of unit redemption price, dividend rate $r$, investment yield $i$, and with $n$ interest periods to maturity Then

$$(1) \qquad P(n) = (1+i)^{-n} + r(1+i)^{-1} + r(1+i)^{-2} + \cdots + r(1+i)^{-n}$$

or

$$P(n) = 1 + \frac{1-(1+i)^{-n}}{i}(r-i)$$

and

$$(2) \qquad \frac{dP}{dn} = \frac{(1+i)^{-n}\log(1+i)}{i}(r-i).$$

Hence $dP/dn$ has the sign of $r-i$. It follows that the premium $P-1$, paid when $r>i$, and the discount $1-P$, for $r<i$, increase with $n$. Similarly it may be shown that the absolute value of $(1-P)/P$ increases with $n$. These well known facts may be stated as follows:

The change in purchase price and the proportional changes in purchase price of a bond, to alter the investment rate from $r$ to $i$ or from $i$ to $r$, increase with increasing length of time to maturity.

It is the purpose of this note to determine whether this holds when the change in investment yield is from $i_1$ to $i_2$, where $i_1 \neq r \neq i_2$.

2. Let $P_1$ be the purchase price of a bond to yield $i_1$, and $P_2$ to yield $i_2$. We shall take $i_1 > i_2$ and $r>0$. Then from (1) it is seen that $P_2-P_1$ is positive for positive $n$ and an analytic function of $n$ for all finite values of $n$. We shall investigate the sign of its derivative,

$$\frac{d}{dn}\left(P_2 - P_1\right) = (1+i_2)^{-n}A_{(n)},$$

where

$$(3) \qquad A(n) = \frac{r-i_2}{i_2}\log(1+i_2) - \frac{r-i_1}{i_1}\left(\frac{1+i_2}{1+i_1}\right)^n \log(1+i_1).$$

For $n=0$ this has the positive value[1]

$$r[i_2^{-1}\log(1+i_2) - i_1^{-1}\log(1+i_1)] + \log(1+i_1) - \log(1+i_2).$$

[1] Note that $x^{-1}\log(1+x)$ decreases for $x>0$ with increasing $x$, since

$$\frac{d}{dx}\ \frac{\log(1+x)}{x} = \frac{x-(1+x)\log(1+x)}{x^2+x^3} = -\frac{\log(1+\xi)}{2\xi+3\xi^2}$$

for some $\xi$, $0<\xi<x$, by Cauchy's formula, and this is negative.

From (3) it is seen that $A(n)$ is a univariant function of $n$ which takes the sign of $r-i_2$ as $n$ approaches infinity. The derivative of $P_2-P_1$ is therefore positive for all values of $n$ if $r>i_2$. For $r<i_2$ the derivative is positive for $n$ less than the root $n_0$ of $A(n)=0$ and is negative for all higher values of $n$. Combining these results with the statements in the introduction we have the theorem:

Theorem: *The change in the purchase price of a bond, to alter the investment rate from $i_1$ to $i_2$, increases with increasing $n$ if either investment rate is less than or equal to the dividend rate. If both investment yields are greater than the dividend rate it will increase to a maximum at $n_0$ and thereafter decrease.*

3. To discover whether the two proportional changes in purchase price, $(P_2-P_1)/P_1$ and $(P_2-P_1)/P_2$, increase or decrease with increasing $n$ it will be sufficient to consider the first alone, since their derivatives have the same sign.

$$\frac{d}{dn}\frac{P_2-P_1}{P_1}=\frac{P_1P_2'-P_2P_1'}{P_1^2}=\frac{P_2^2}{P_1^2}\frac{d}{dn}\frac{P_2-P_1}{P_2}.$$

From (2) it follows that this derivative is positive if $i_1>r_2>i_2$. To discuss the other cases we write

$$\frac{d}{dn}\frac{P_2-P_1}{P_1}=\frac{(1+i_2)^{-n}A(n)}{P_1^2},$$

where

$$i_1i_2A(n)=\{r-(r-i_1)(1+i_1)^{-n}\}(r-i_2)\log(1+i_2)$$
$$-\{r-(r-i_2)(1+i_2)^{-n}\}(r-i_1)(1+i_2)^n(1+i_1)^{-n}\log(1+i_1).$$

$A(n)$ has the positive value

$$r\{i_2^{-1}\log(1+i_2)-i_1^{-1}\log(1+i_1)\}+\log(1+i_1)-\log(1+i_2)$$

for $n=0$, and the sign of $r-i_2$ as $n$ approaches infinity. Hence the derivative of $(P_2-P_1)/P_1$ is positive for $n=0$. It is also positive as $n$ approaches infinity if $r>i_2$ but negative if $r<i_2$. To discover its sign for values in between we write it in the form

$$\frac{d}{dn}\frac{P_2-P_1}{P_1}=\frac{(1+i_1)^{-n}(1+i_2)^{-n}}{i_1i_2P_1^2}B(n),$$

where

$$B(n)=\{r(1+i_1)^n-(r-i_1)\}(r-i_2)\log(1+i_2)$$
$$-\{r(1+i_2)^n-(r-i_2)\}(r-i_1)\log(1+i_1).$$

$B(0)$ is positive and its derivative,

$$dB/dn=r(1+i_2)^n\log(1+i_1)\log(1+i_2)C(n),$$

where

$$C(n) = \left\{\left(\frac{1 + i_1}{1 + i_2}\right)^n - \frac{r - i_1}{r - i_2}\right\}(r - i_2),$$

has the sign of $C(n)$. The latter is a univariant function positive for $n=0$ and has the sign of $r-i_2$ as $n$ approaches infinity. For $r>i_1>i_2$, $C(n)$ is positive for all $n$. For $r<i_2<i_1$ it is positive for $n<n_1$, the root of

$$(r-i_2)\ (1+i_1)^n=(r-i_1)\ (1+i_2)^n,$$

and is negative for $n>n_1$. In the first case $B(n)$ is positive for all positive $n$. In the second case it is positive and rises to a maximum at $n=n_1$, thereafter decreasing to zero at $n_2$, the root of $B(n)=0$. For higher values of $n$ it is negative. In the first case the derivative of $(P_2-P_1)/P_1$ is positive for all positive values of $n$. In the second case it is positive for all values such that $n>n_2$ and is negative for all higher values of $n$. We have as before the theorem:

Theorem: *The proportional changes, $(P_2-P_1)/P_1$ and $(P_2-P_1)/P_2$, in the purchase price of a bond, to alter the investment yield from $i_1$ to $i_2$, increase with increasing n if either yield is less than or equal to the dividend rate. If both investment yields are greater than the dividend rate these proportional changes increase to a maximum at $n=n_2$ and thereafter decrease.*

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## ON THE GEOMETRY OF CLOCKS

By J. M. FELD, Columbia University

In a paper on *Homographic circles or clocks* by L. Hoffman and E. Kasner,[1] was introduced the term *homographic clocks* for circles whose points are in homographic correspondence with the points of the unit circle. If $t=e^{i\theta}$ ($\theta$ real) represents the unit circle on the Gaussian plane, then any homographic clock $Z$ is given by the equation

$$z = (A + Be^{i\theta})/(C + De^{i\theta}).$$

The clock is termed positive or negative according as $z$ moves on the circle $Z$ in the same sense that $t$ moves on the unit circle or not. Thus the clocks $Ae^{i\theta}$ and $Be^{-i\theta}$ are positive and negative, respectively.

The object of this paper is to study the geometry associated with a sum of two clocks $Ae^{i\theta}$ and $Be^{-i\theta}$. Let

$$z = Ae^{i\theta} + Be^{-i\theta}, \tag{1}$$

where $z=x+iy$ and $A$ and $B$ are constants. By separating the real and imaginary parts in (1) we get

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[1] Bulletin of the American Mathematical Society, vol. 34 (1928), p. 495.

$$x = (a_1 + b_1)\cos\theta - (a_2 - b_2)\sin\theta,\quad y = (a_2 + b_2)\cos\theta + (a_1 - b_1)\sin\theta,$$

where $A = a_1 + ia_2$ and $B = b_1 + ib_2$. Letting $a_1 + b_1 = p$, $a_2 - b_2 = q$, $a_2 + b_2 = r$, $a_1 - b_1 = s$, the Cartesian equation of the curve (1) becomes

$$(r^2 + s^2)x^2 - 2(pr - qs)xy + (p^2 + q^2)y^2 = (ps + qr)^2.$$

Thus (1) represents an ellipse having its center at the origin.

From the form of equation (1) we obtain a method for constructing an ellipse. Let $A$ and $B$ be any two vectors, $OA$ and $OB$, respectively. We form the linkage $OAB$ by adding vector $B$ to $A$. As $OA$ rotates counterclockwise about $O$, the link $AC$ rotates clockwise, and at the same rate as $OA$, about $A$. The point $C$ generates the ellipse.

Let $|z| = m$ and $\bar{z} = x - iy$. From (1) we get

$$m^2 = (Ae^{i\theta} + Be^{-i\theta})(\overline{A}e^{-i\theta} + \overline{B}e^{i\theta})$$

or

$$m^2 = A\overline{A} + B\overline{B} + A\overline{B}e^{2i\theta} + \overline{A}Be^{-2i\theta}. \tag{2}$$

For maximum and minimum $m$ we must have

$$e^{4i\theta + 2\pi ik} = \overline{A}B/A\overline{B}\ (k \text{ an integer})$$

or

$$\theta = \tfrac{1}{2}\pi k + \tfrac{1}{2}(\arg B - \arg A). \tag{3}$$

When $k$ is zero or even, $m$ is a maximum; and when $k$ is odd, $m$ is a minimum. Replacing $\theta$ in (2) by $\frac{1}{2}(\arg B - \arg A)$ and by $\frac{1}{2}\pi + \frac{1}{2}(\arg B - \arg A)$ we get the lengths of the semi-major and semi-minor axes of the ellipse, respectively. Letting $a$ and $b$ represent the semi-major and semi-minor axes we find

$$a = |A| + |B|,\quad b = |A| - |B|.$$

If $\phi$ is the angle that the major axis makes with the axis of reals, we get from (3)

$$\phi = \arg A + \tfrac{1}{2}(\arg B - \arg A) = \tfrac{1}{2}(\arg A + \arg B).$$

The major axis therefore bisects the angle formed by the vectors $OA$ and $OB$.

The distance of the foci from the center $O$ is given by

$$c = (a^2 - b^2)^{1/2} = 2|AB|^{1/2}.$$

If in (1) $A$ and $B$ are regarded as parameters and if we make $AB = C$, where $C$ is a constant, (1) represents a family of confocal ellipses since

$$\tfrac{1}{2}(\arg A + \arg B) = \tfrac{1}{2}\arg C \quad \text{and} \quad 2|AB|^{1/2} = 2|C|^{1/2}.$$

## QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

*The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.*

### ON SOME LOCI CONNECTED WITH THE ORTHOPOLE-GEOMETRY

By R. GOORMAGHTIGH, Bruges, Belgium

In the March, 1930 issue of this Monthly[1] was published a paper by O. J. Ramler on *The orthopole loci of some one-parameter systems of lines referred to a fixed triangle.*

Several properties obtained in that article have already been mentioned[2] and it may be interesting to recall, with some notes on the literature of the subject, a few other theorems about the loci considered.

1. *Locus of the orthopoles of the Simson lines.* In *Mathesis* (1914), p. 152, we have proved the following theorem:

*If $TT'$ is a diameter of the circumcircle of a triangle $A$, $B$, $C$, the orthopole of the Simson line of $T$ is the projection of the orthocenter $H$ on the Simson line of $T'$.*

Hence, the locus of the orthopoles of the Simson lines of a triangle is the *pedal curve of the Steiner deltoid for the orthocenter H* (See our question No. 17927 in the *Educational Times*, 1915, p. 39, and solutions by C. E. Youngman and R. F. Davis, *ibid*, 1915, p. 164).

2. *Locus of the orthopoles of the tangents to the circumcircle.* Neuberg has proved in his *Mémoire sur les Projections et Contre-projections d'un triangle fixe* 1890 p. 76, that the locus is the Steiner deltoid.

More generally, when a straight line $L$ remains tangent to a circle concentric to the circumcircle, the locus of the orthopole of $L$ is a trochoidal curve (L. Poli; Question 2490 in *Nouvelles Annales de Mathématiques*, 1920, p. 160; our proof is given in a paper presented by V. Thébault to the *Société Scientifique de Bruxelles*, 1926).

3. *Locus of the orthopoles of the straight lines passing through a fixed point P.* The fact that the conic, locus of these orthopoles, is always an ellipse is proved by A. A. Krishnaswami Ayyangar, in his paper *A simple exposition of the allied properties of the Simson line, the Steiner envelope, and the orthopole* (*Educational Review*, Madras, October, 1924).

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[1] Vol. 37 (1930), pp. 130–136.

[2] R. Goormaghtigh, *On the properties of the orthopole*, Tohoku Mathematical Journal, vol. 30 (1926), pp. 77–125.

The barycentric equation of the orthopolar conic of $P$, in the pedal triangle of $P$, is

$$a^2\alpha^{-1} + b^2\beta^{-1} + c^2\gamma^{-1} = 0,$$

$a, b, c$ being the length of the sides $A, B, C$.
This equation is given in Neuberg's article *Sur les cercles polaires relatifs à un triangle fixe* (*Bulletin de l'Académie royale de Belgique, classe des Sciences*, July-August, 1910).

The equation shows that, *if the pedal triangle of P is projected on a plane into a triangle similar to A B C, the orthopolar conic is projected into a circle* (see article in the *Tohoku Mathematical Journal, 1925*.

The properties about the length of the axes of the orthopolar conic are also to be found in Ayyangar's paper; it contains the following determination of the directions of the axes: *they are perpendicular to the Simson lines of the extremities of the circumdiameter passing through P.*

We have further shown (*Nouvelles Annales de Mathématiques*, 1914, p. 121) *that the orthopolar conics of a triangle are all inscribed in the Steiner deltoid* and (*Mathesis*, 1914, p. 121) that *the orthopolar conic of a point P is inscribed in the triangle when P is the image of H through the circumcenter.*

4. Finally it may be interesting to note another locus connected with the subject:

*When a straight line L passes through a fixed point, the locus of the image of the orthopole of L, through L, is a limaçon.*

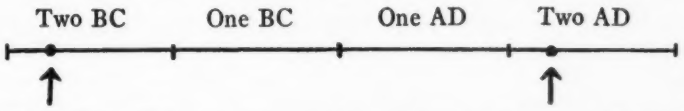
For a proof see our article *Sur l'orthopôle et certains limaçons de Pascal associés au triangle*, in *Nouvelles Annales de Mathématiques*, 1918, p. 242.

## A Letter to the Editor

By Eward S. Smith, Denver, Colorado

In the Monthly for March (1930) I find on page 150 what seems to me an error, in the selection of the dates 570, 470 and 370 B.C. as dates for which the present year is a multicentennial.

When Dionysius Exiguus devised an improved system of chronology, he could have saved future historians and computers some annoyance if he had designated the year following B.C. one as the year zero. He was possibly not acquainted with zero and its helpful properties; anyway the year after B.C. one was A.D. one. It is easy to see, where the two years concerned are near together, one being B.C. and the other A.D., that in order to compute the interval, the two numbers must be added together and the sum diminished by one.




This diagram shows that the interval between say, the Ides of March, B.C. 2, and the Ides of March A.D. 2, is not $2+2$ but $2+2-1=3$ years.

Unless there is some pertinent consideration that I have failed to notice, we should take the year 1931 instead of the present year as the correct time to celebrate the multi-centennial of events that occurred in B.C. 570, 470, 370, 270.

Thus, if Anaximenes were still living on Earth (he is probably in a better place now) he would in 1931 be $570+1931-1=2500$ years old, and would undoubtedly still be studying (and perhaps teaching) mathematics.

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## RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of the New York

*All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y. and not to any of the other editors or officers of the Association.*

### NEW BOOKS RECEIVED

*Advanced Mathematics for Students of Physics and Engineering.* By D. Humphrey. Oxford University Press, 1929. Part I, viii+120 pages. Part II, viii+176 pages. $4.25.

*Plane and Spherical Trigonometry.* By R. D. Carmichael and E. R. Smith. Boston, Ginn and Co., 1930. x+200 pages. $1.60.

*Analytic Geometry.* By D. R. Curtiss and E. J. Moulton. Boston, D. C. Heath and Co., 1930. xiv+338+18 pages.

*The Size of the Universe.* By Ludwik Silberstein. Oxford University Press 1930. vii+216 pages. $3.50.

*An Introduction to the Geometry of n Dimensions.* By D. M. Y. Sommerville. E. P. Dutton, 1930. xvi+196 pages. $3.90.

*Einführung in die Nicht-Euklidische Geometrie.* By Hans Mohrmann. Leipzig, Akademische Verlagsgesellschaft M. B. H., 1930. xii+128 pages.

*Statistical Methods for Research Workers.* By R. A. Fisher. Third edition, Revised and Enlarged. London, Oliver and Boyd, 1930. xvi+284 pages, 6 tables 15s.

*Solid Analytical Geometry and Determinants.* By Arnold Dresden, New York, John Wiley and Sons, 1930. x+310 pages. $3.00.

*Algebra for Junior and Senior High Schools.* By J. W. Calhoun, E. V. White, T. Mc N. Simpson, Jr. Richmond, Johnson Publishing Company, 1930. xii+486 pages.

*Schriften-Herausgegeben von der Königlichen Böhmischen Gesellschaft der Wissenschaften.* By Bernard Bolzano. Vorrede von Dr. Karel Petr. Band I, *Funktionentheorie.* Prag, 1930. xx+184 pages; Anmerkungen, 24 pages, index and errata vi pages.

*Statistical Tables and Graphs.* By Bruce D. Mudgett. Boston, Houghton Mifflin Co., 1930. viii+194 pages. $1.75.

*Grundlagen der Analysis.* By E. Landau. Leipzig, Akademische Verlagsgesellschaft M. B. H., 1930, xiv+134 pages.

*Elementargeometrie der Ebene und des Raumes.* By Max Zacharias. Berlin, de Gruyter, 1930. 252 pages. Rm 13.50.

*Darstellende Geometrie.* ii. *Perspektive Ebene Gebilde, Kegelschnitte.* By Robert Haussner. Sammlung Göschen 143. Berlin, de Gruyter, 1929. 168 pages.

*Élèments de Mathèmatiques financières. Opèrations à long terme.* By R. Thery. Paris, Vuibert, 1930. viii+92 pages.

*Calculus.* By E. J. Miles and J. S. Mikesh. New York, The McGraw-Hill Book Company, 1930. xiv+638 pages. $3.75.

*Projective Geometry.* By J. W. Young. The Carus Mathematical Monographs, No. 4. Chicago, The Open Court Publishing Co., 1930. ix+185 pages. $2.00.

*The Adjustment of Errors in Practical Science.* By R. W. M. Gibbs. Oxford University Press, 1930.

*Source Book in Astronomy.* By Harlow Shapley and Helen E. Howarth. New York, The McGraw-Hill Book Co., 1929. xvi+412 pages.

*Differential Geometry of Three Dimensions.* By C. E. Weatherburn. Volume II, Cambridge University Press, 1930. xii+240 pages. $4.25.

*The Theory of Approximations.* By Dunham Jackson. New York, The American Mathematical Society. Colloquium Publications, vol. xi, 1930. viii+178 pages.

*Differential Equations.* By F. R. Moulton. New York. The Macmillan Company, 1930. xvi+396 pages. $5.50.

*Gewöhnliche Differentialgleichungen.* By G. Hoheisel. Second edition, enlarged. Sammlung Göschen 920. Berlin, de Gruyter, 1930. 159 pages. Rm. 1.80.

## REVIEWS

*Special curves.* By R. C. Archibald, Encyclopaedia Britannica, vol. 6, 1929, pp. 887–899.

This remarkable article contains the description of some 120 curves. The reader is greatly aided by an alphabetical list, followed by a systematic discussion arranged in sections in this order: plane algebraic curves according to their degrees, plane transcendental curves, general classes of plane curves, and curves of double curvature. In addition to short historical notes and a wealth of bibliographical information concerning these curves, there is a concise but very clear account of the fundamental geometric properties of the curves in question (with figures), together with their equations in rectilinear, polar, and other coordinate systems. An attentive reader will find here many interesting details never before mentioned in the literature. For instance, it appears that the curves resulting from addition of two simple harmonic motions along two orthogonal axes, which usually are known under the name of Lissajous curves, were actually studied first by Nathaniel Bowditch, author of the well known book on navigation, some 40 years before Lissajous. Although no one would expect completeness in a brief monograph for the Encyclopaedia, there are still some interesting classes of curves which, perhaps, should have been mentioned here. Such are, for instance, the class of curves of constant width

studied since Euler, and certain curves introduced by Tchebyscheff. There is no doubt, however, that R. C. Archibald has contributed a reference source that will prove in many instances more useful and more reliable than some of the large treatises on the subject mentioned at the end of the article.

J. TAMARKIN

*Le théorème de Picard-Borel et la théorie des fonctions méromorphes.* By Rolf Nevanlinna. Borel Series of Monographs. Gauthier-Villars et Cie, Paris, 1929. vii+174 pages.

The main object of this book is the study of the distribution in the $x$-plane of the roots of the equation $f(x)=z$ for all (finite or infinite) values of $z$, where $f(x)$ is a transcendental function of $x$ meromorphic for all (finite) values of $x$. A number of the results obtained are extensions of classical results of Hadamard, Borel, and others for entire functions. Some of the theorems of the book have not previously appeared in the literature; others have been proved quite recently by such writers as Lindelöf, Valiron, F. Nevanlinna, and of course the author himself.

If $f(x)$ is such a function as is considered above, and if the equation $f(x)=z$ has no roots $x$ for a particular $z$, that value $z$ is called an *exceptional value*, and the classical theorem of Picard asserts that there are at most two exceptional values. Nevanlinna's first fundamental theorem (1925) studies not merely the distribution of possible roots $x$ of the equation $f(x)=z$ but involves likewise the average convergence, for $|x|$ becoming infinite, of $f(x)$ to the value $z$, and it asserts that if suitable measures of these two quantities are considered together, *there are no exceptional values*. Or, as Nevanlinna expresses it, the *total affinity* of the function $f(x)$ for the value $z$ is independent of $z$.

We must refrain for lack of space from going into detail concerning the results established. The notion of *order* of a meromorphic function is introduced and the analogues of the results of Hadamard for entire functions proved. This leads to the representation of such a function by canonical products. The second fundamental theorem gives more detailed information about the distribution of the roots $x$ of the equation $f(x)=z$, and has varied and beautiful applications. Finally there are studied systems of meromorphic functions connected by a linear relation and functions assumed meromorphic merely in a finite circle.

The treatment throughout the book is clear and elegant, and many of the theorems are of great interest and beauty. The results are in the main established by the systematic use of the formula of Jensen in the form:

$$\frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log} |f(\rho e^{i\theta})|\, d\theta + \int_0^{\rho} \frac{n(r,\infty)-n(0,\infty)}{r}dr + n(0,\infty)\log\rho$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log} \left|\frac{1}{f(\rho e^{i\theta})}\right| d\theta + \int_0^{\rho} \frac{n(r,0)-n(0,0)}{r}dr$$
$$+ n(0,0)\log\rho + \log|c_\lambda|$$

where $\log^{+} \alpha$ is defined as $\log \alpha$ or zero according as $\alpha \geqq 1$ or $0 \leqq \alpha < 1$, where $n(r, 0)$ and $n(r, \infty)$ are the numbers of zeros and of poles of $f(x)$ for $|x| \leqq r$, and where in the neighborhood of the origin we have

$$f(x) = c_\lambda x^\lambda + c_{\lambda+1} x^{\lambda+1} + \cdots, \quad c_\lambda \neq 0.$$

We can recommend to anyone who is interested in the advances now being made in the theory of functions of a single complex variable, or who wants an indication of some of the work done in the past decade, or who would have presented to him some of the important unsolved problems in the theory, this book by R. Nevanlinna.

J. L. WALSH

*The Calculus.* By Hans H. Dalaker and Henry E. Hartig. McGraw-Hill Book Company, New York, 1930. viii+254 pages. $2.25

This textbook is designed for a first course in calculus and follows to a large extent the tradition of Granville and others. The authors say: "Especial effort has been made to guide the student to an understanding of basic principles and inner meanings." The first two chapters show a serious attempt to justify this statement. In them are discussed very carefully matters of notation and the subjects of limits and continuity. One feels that the average student can master this material without undue difficulty and thus lay a firm foundation for the study of the calculus.

In Chapter III the derivative is defined. Here, following the Granville tradition, the symbol $dy/dx$ is used. Then the two usual interpretations of the derivative are given, viz, *the derivative as a rate of change*, and *the derivative as the slope of a curve*. The former interpretation usually causes the student a great deal of trouble; and one is inclined to think that the explanation given on page 12 will carry little meaning to him. As to the latter interpretation one must remember that most students studying calculus have not studied tangents except in the case of the circle. It would seem that the explanation of the slope would be more convincing if the definition of the tangent to any curve were given. The students will probably require a great deal of help from the teacher before they will have mastered these two interpretations of the derivative.

One wonders why the symbol $dy/dx$ is used for the derivative at the beginning of the course when it causes so much uncertainty in the minds of so many students. They fail to realize that the derivative is the limit of a ratio, and think that in some unknown way $\Delta y$ has become $dy$ and $\Delta x$ has become $dx$. Certainly there are many teachers who feel that the symbols $D_x y$, $y'$, and $f'(x)$ produce better results.

The first ten chapters of the book are devoted largely to a mastery of the technique of differentiation. Chapters XI–XV deal with applications of differentiation to geometry and physics. There is a wealth of well-graded exercises

in these chapters. However, one feels that the problems in the applications to geometry can be handled to a large extent mechanically by the student. Chapter XVI deals with the *law of the mean and indeterminate forms* in the manner that is usually found in textbooks in calculus.

After a detailed study of differentiation, integration is studied in great detail. A student who works through the lists of exercises on indefinite integration will have a good knowledge of the technique of the subject. Following the drill on the indefinite integral, the authors define the definite integral. The treatment of this important subject is excellent. The basic principles of definite integration are brought out in such a way that the student should be able to apply them readily to a large variety of problems. One is impressed by the excellence of the figures in the chapter on multiple integrals. The treatment of this subject has been kept to an irreducible minimum; and one is inclined to believe that it could have been profitably expanded at the cost of curtailing some of the earlier material on technique.

In addition there are a brief chapter on partial differentiation and an excellent one on infinite series. The book closes with chapters on *Expansion of Functions*, *Hyperbolic Functions*, *Change of Variables*, and *Approximate Integration*.

One is inclined to think that the book will be widely used. It has the virtues of the old Granville text, and also many that the earlier book did not have. But in spite of the efforts that the authors have made to stress fundamental principles, a large percentage of the students who use this book will probably acquire little more than a knowledge of the technique of differentiation and integration.

One wonders if the writers of textbooks of calculus are losing an opportunity to serve the natural sciences. Men working in chemistry and biology are crying for help in mathematics. A dean of a medical school says that biologists should understand the fundamental principles of calculus, but that they never need the mastery of the technique of differentiation and integration which is required by most teachers of mathematics. Is a knowledge of so much technique necessary even for engineers and physicists? The long lists of exercises on differentiation remind one of those on factoring in textbooks in elementary algebra of thirty years ago. One ventures to hope that in the near future a textbook in calculus will appear which will stress fundamental principles and in which technique is reduced to a minimum. Isn't it possible that such a book will meet the needs of biologists, chemists, engineers, physicists, and mathematicians better than the books which give so much attention to technique?

W. G. SIMON

*Calculus*. By Egbert J. Miles and James S. Mikesh. The McGraw-Hill Book, New York, 1930. xii+638 pages. $3.75

The main thesis of the authors is that the concept of rate of change should be given the major rôle in a first course in the calculus, and that everything can

be and should be developed around rate of change as a unifying principle. They believe that many of the things that are usually done are bad to do; they know exactly what they want to do, and they proceed to do it throughout 638 pages. No change from tradition is too great to maintain rate of change in its central position. Thus the differential is practically eliminated, integration is anti-differentiation, and no use is made of the integral as the limit of a sum. Whether or not it is advisable to emphasize rate of change is debatable—and although it is of no moment, the reviewer believes it is not desirable— but surely such far-reaching consequences as these require very careful consideration.

Limits, rate of change, differentiation and applications, integration and applications occupy the first thirteen chapters (263 pages). In these chapters differentiation is confined to algebraic functions, and the scope of the integration is thus grestly restricted. Chapter XIV on "Appoximations" contains among other things Maclaurin's and Taylor's series. An unexpected chapter (XV) on functions of several variables contains not only a brief treatment of partial differentiation, but also exact derivative (differential) equations, and even line integrals. In Chapter XVI the exponential and logarithmic functions are introduced, the discussion is detailed, and the many physical and chemical applications lead rather naturally to the consideration of first order linear differential equations. The trigonometric functions are introduced in Chapter XVII and treated at such length that the chapter runs to 114 pages. Section 104, "Solution of Triangles," is exactly that; the reason for its insertion in a text on the calculus is quite unknown to the reviewer.

The maintenance of rate of change as a unifying principle leads the authors to make substantial changes in notation. No $d$'s nor deltas are introduced, the derivative of $f(x)$ when $x=x_1$ is the constant $f'(x_1)$, the derivative function is $f'(x)$, the integral of $f(x)$ is $\int f(x)$. In differentiation the "delta-process" has been completely eliminated. In this connection the authors say: " . . . . the student feels he is continuing work already begun in algebra, and his attention is focused on the principle involved rather than upon the manipulatory processes." We do not feel that this claim is justified. Specifically we refer the reader to pages 38, 39, and 81 where the derivative of $u^n(x)$ is obtained. Integration without the differential notation is by no means impossible, but it does seem to involve added difficulties for both student and instructor. The excessive use of the solidus is displeasing, typographically, and this and other usages make for errors which are all too numerous throughout the book.

We have noted the inclusion of functions of several variables and of differential equations. If we leave these topics aside, the scope of the book, despite the six hundred odd pages, is distinctly less than that of the average American text. Curvature, evolutes, singular points, centers of gravity, moments of inertia, and mechanics are entirely omitted. One might ask how the authors employ so many pages. Nearly 40% of the space is occupied by illustrative examples, worked out in complete detail and with full explanations. The probblems, nominally 1100 in number, actually total about 2100 if distinct parts

be considered separate problems. The uniform simplicity of these disturbs us. With the exception of two theoretical problems (#3 on page 245, and #13 on page 475) there is not a problem in the entire book which constitutes a real challenge to a good sophomore. The text is full, accurate, and thorough, there is careful motivation, and the explanations are unusually detailed. We appreciate fully that there is honest workmanship, that this text adheres to definite aims, and that it endeavors to set up standards of accurate thought. Yet we must add our personal conviction that, pedagogically, this is not a good book. There is too much detailed explanation, too much illustration; and the student who studies faithfully an assignment of seven pages will have little energy and no inclination to tackle problems for himself. There is little opportunity here for the student to develop initiative and mathematical robustness. In our judgment, the authors fail to understand either the weaknesses or the strengths of the American sophomore.

B. H. BROWN

*Enciclopedia delle Matematiche Elementari.* L. Berzolari, G. Vivanti, D. Gigli (general editors). Volume 1, Part 1. Milan, Ulrico Hoepli, 1930, 9+450 pages.

The present "Part 1", is to be followed at once by "Part 2," announced as already under press. Volume 1 is devoted to analysis, and is to be one of three volumes constituting this encyclopedia. The second volume is to deal with geometry and the third with applications of mathematics, history of mathematics and didactic questions. The present part comprises seven independent articles together with indices etc., namely: I. Logic, by Alessandro Padoa; II. General arithmetic, by Duilio Gigli; III. Practical arithmetic, by Ettore Bortolotti and Duilio Gigli; IV. Theory of numbers and indeterminate analysis, by Michele Cipolla; V. Progressions, by Aldo Finzi; VI. Logarithms, by Aldo Finzi; VII. Mechanical calculation, by Giuseppe Tacchella.

This is the first work of this sort in the Italian language and the editorial commission has approached its task with an enthusiasm that suggests not merely service in a needed field, but also patriotic fervor, as seen throughout the preface dated "1929 (A. VII)." The form and subject matter may well be contrasted with that of Pascal's "Repertorium" (not mentioned in this work) whose expanded German editions are proving to be of the highest service for ready reference. The treatment here used is intended to meet the demand of teachers and younger students who do not have ready access to basic sources and the material to be covered is intended to be roughly that touched by the end of the first two years of university courses.

Any elaborately detailed and critical analysis of each of the seven condensed treatises which are here offered to the reader will be inappropriate. The American teacher will find most of the material elsewhere in a language more familiar to him. Despite excellent historical references, completeness in this book is not promised. The value of the work cannot consist in its completeness, nor

in its bringing to light unsuspected historical sources. It must be judged by the degree to which it renders available to college and secondary school teachers the important elements touching upon the basic mathematical courses, and the good judgment with which topics from so vast a possible field are selected and woven into smoothly flowing expositions. The authors have refrained from that hint of superiority that might be suggested by the familiar phrase, "vom höheren Standpunkte aus" and incidentally have missed much of the lively interest for the graduate student that characterizes Klein's work of a somewhat similar purpose (*Elementarmathematik vom höheren Standpunkte aus*, Third Edition, 1924–1928). One ought rather to compare this work with the H. Weber and J. Wellstein, *Encyklopädie der elementar-Mathematik* (Vol. 1, 4th Edition 1922, Vol. 2, 2nd Edition 1907. Vol. 3, 1st Edition 1907).

The first section (of 79 pages) is a serious study of the formal elementary aspects of modern logic as used by Peano and later by Russell and others. This section is not referred to in later parts and would seem to be somewhat out of keeping with the general tenor of the work, despite the fact that mathematics uses logic and to a considerable extent is logic. Controversial questions and all references to paradoxes are suppressed, but the reader may agree with the reviewer in feeling that the theories have not yet been subjected to that intensive critical analysis by many independent workers which is the only practicable test of scientific stability. The astonishing number of 143 independent (?) axioms are listed consecutively.

The section on general arithmetic (covering 130 pages) deals with what is often called the number-concept, starting with finite classes and natural numbers and concluding with the transcendence of $e$ and $\pi$. The treatment is excellent although necessarily far from novel. It may be too abstract to suit many college teachers in this country, but would be worth any effort required to understand it.

The third section devoted to "practical arithmetic" is what many persons might call the abstract theory of arithmetic. Apart from some interesting historical topics the section deals in a logical but thoroughly elementary way with the fundamental operations of arithmetic, and the proofs incident thereto (assuming much prior logical foundation). Questions concerning mensuration, percentage and other topics of commercial arithmetic find no place.

The fourth section treats of certain topics in elementary number theory. Apart from a very brief preliminary discussion, two main divisions deal respectively with linear congruences (Fermat-Euler theorem, the Gaussian, Wilson's theorem, Euclid's algorithm, etc.), and with higher congruences (roots of unity, quadratic congruences, etc.). In no other section does one find so many familiar topics omitted, but what has been retained shows an unusual attempt at continuity. The treatment is simple, interesting and in connection with topics treated reasonably comprehensive.

The fifth section, dealing with progressions, seems in strange company. It discusses only the traditional arithmetic, geometric, and harmonic series and

some of the most elementary extensions of the first two to figurate numbers. The use of $x/\!\!\wedge y$ to denote $x^y$ is the most noticeable single item of interest. The section does, of course, form a natural bridge from linear congruences to the introduction of logarithms.

The sixth section, on Logarithms, is more than usually effective and covers material ordinarily assumed to be too detailed for elementary texts and too special for advanced mathematical courses.

The seventh section, on computing machines, suffers only from the common difficulty incidental to merely talking about mechanical devices whose virtues lie in their concrete utility. While one machine actually used would outweigh in significance any number of mere descriptions the reader is at least prepared to admit that there are many machines and for divers purposes.

The paper is poor, the binding flimsy, the figures cheaply drawn, and typographical mistakes not unknown (The table of contents lists "Errata, pag. 415." It should read "P. 451."). But throughout the presentation is scholarly, the emphasis well-placed, and the language simple, connected and interesting. Whether the completed work will prove eventually more attractive to American readers than the somewhat more conventional "Weber-Wellstein," will remain to be seen.

ALBERT A. BENNETT

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## PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

*Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.*

### PROBLEM FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3441. *Proposed by Solomon Kullback, Brooklyn, New York.*

In a given triangle, inscribe an equilateral triangle having a given point, $P$, on one of its sides. Suggested by problem No. 3405.

3442. *Proposed by the late F. P. Matz*

Solve the equation $d/dw(dw/dr+2w/r)=0$.

3443. *Proposed by Warren A. Rees, Houston Junior College.*

To construct a quadrilateral, given the four feet of the perpendiculars from the point of intersection of the diagonals upon the four sides.

3444. *Proposed by Frank Morley, Johns Hopkins University.*

In an inversive plane, the general self-conjugate equation, $f(x,\bar{x})=0$, of degree three in $x$ and $\bar{x}$ defines a bi-cubic curve, $c$. Since any circle has with such a curve six common points (intersections or common unique pairs) there are contact circles, touching thrice. There are, it is known, 120 contact circles. If we take three of these, the 9 points of contact either lie on a biquadratic $b$, or they do not. When they do, the circles are tied (or syzygetic); and the curve, $b$, meets $c$ in the points of contact of a fourth circle, so that the circles are tied in sets of four. Prove that the four circles of a set touch a circle.

3445. *Proposed by Mannis Charosh, New Utrecht High School.*

If $p$ is odd and greater than 1, prove that

(a) $$1^2\cdot 3^2\cdot 5^2\cdots(p-2)^2\equiv(-1)^{(p+1)/2}\bmod p;$$

(b) $$2^2\cdot 4^2\cdot 6^2\cdots(p-1)^2\equiv(-1)^{(p+1)/2}\bmod p.$$

3446. *Proposed by J. Rosenbaum, Milford, Conn.*

Given $n$ points, $A_1, A_2, \ldots, A_n$, and $n$ constants $a_1, a_2, \ldots, a_n$, show how to locate a point $X$ such that the vectors

$$a_1\cdot XA_1,\ a_2\cdot XA_2,\cdots,a_n\cdot XA_n$$

shall form a closed polygon.

This is a generalization of problem 3395 (November, 1929).

Here the point $X$ is the center of mass of the $n$ masses $m_1, m_2, \ldots, m_n$ placed at $A_1, A_2, \ldots, A_n$ and proportional to $a_1, a_2, \ldots, a_n$.

3447. *Proposed by Vladimir F. Ivanoff, San Francisco, California.*

Prove that

$$2R = R_x\cos\alpha + R_y\cos\beta + R_z\cos\gamma,$$

where $R$ is the radius of curvature of a given curve at the point $(x,y,z,)$; $R_x$, $R_y$, $R_z$ are the radii of curvature of the projections of this curve on the co-ordinate planes, $YOZ$, $XOZ$, and $XOY$ at the points $(y_1, z_1)$, $(x_1, z_1)$ and $(x_1, y_1)$, respectively; and $\alpha, \beta, \gamma$ are the angles between $R$ and $R_x$, $R_y$, $R_z$, respectively.

## SOLUTIONS

3366. [1929, 168]. *Proposed by Otto Dunkel, Washington University.*

Given the two equations

$$x^{r+2} - bx^{r+1} \pm l(x-a) = 0, \qquad 0 < a < b,\ l > 0,$$

where $r$ is any positive number, let $r_1$ and $r_2$ be the smallest and largest positive roots of the second equation (with the lower sign). Prove that all of the roots of the two equations except $r_1$ and $r_2$ lie within the circular ring with radii $r_1$ and $r_2$ about the origin as center.

Let $OA=a$, $OB=b$, $OR_1=r_1$, $OR_2=r_2$, and let $R_1'$ $R_2'$ be the harmonic conjugates of $R_1$, $R_2$ with respect to $A$ and $B$. Show that neither equation has positive roots on the segments $R_1R_1'$, $R_2R_2'$ except the two roots of the second equation $r_1$, $r_2$.

*Solution by the Proposer.*

Setting $f(x)=x^{r+2}-bx^{r+1}-lx+la$, we see that $f''(x)=(r+1)x^{r-1}[(r+2)x-br]$ vanishes only once for positive values of $x$. Hence $f'(x)$ decreases from the negative value $-l$ at $x=0$ to a minimum at $x=br/(r+2)$; it then increases from that point on. Hence $f'(x)$ vanishes once and only once for positive values of $x$, and therefore $f(x)$ cannot have more than two positive roots. Since $f(0)=la$, $f(a)=-a^{r+1}(b-a)$, $f(b)=-l(b-a)$, $f(\infty)=\infty$, it follows that $f(x)$ has precisely two positive roots $r_1$ and $r_2$ such that $r_1<a<b<r_2$.

In the plane of the complex variable $z$ designate by $A$ and $B$ the points determined by $z=a$ and $z=b$ on the positive real axis; and consider the family of circles with centers on the real axis and orthogonal to the circle with the diameter $AB$. Any point $P$ on one of the circles $(C_\lambda)$ of the family with the center $C_\lambda$ determines the ratio $AP/BP=\lambda$ which is constant as the point moves on this circle. For $\lambda=0$ we have the null circle at $A$; and, as $\lambda$ increases, $C_\lambda$ moves to the left of $A$, the circles increase in size enclosing each of the preceding circles without contact or intersection. When the center is at $-\infty$, $\lambda=1$ and the circle has an infinite radius; it is the straight line perpendicular to $AB$ at its mid-point. As $\lambda$ increases from 1, $C_\lambda$ moves from $+\infty$ towards $B$, the circles decrease in size, and each lies within each of the preceding circles. This continues until $\lambda=\infty$ and we then have the null circle at $B$.

Designate by $R_1$ and $R_2$ the points determined by the two roots above, $r_1$ and $r_2$. Then $R_1$ and $R_2$ determine two circles of the family with the ratios $\lambda_1$ and $\lambda_2$, the first with the diameter $R_1R_1'$ containing $A$ and the second with the diameter $R_2'R_2$ containing $B$. Let $z$ be a root of either one of the two equations of the problem and suppose that $|z|<r_1$. Let $Z$ be the point determined by $z$. Then

$$\frac{|z|^{r+1}}{l}=\left|\frac{z-a}{z-b}\right|=\frac{AZ}{BZ}.$$

Consider the circle of the family through $Z$; for this circle $\lambda=|z|^{r+1}/l<{r_1}^{r+1}/l$. But since $r_1$ is a root of $f(x)$,

$${r_1}^{r+1}/l=(r_1-a)/(r_1-b)=AR_1/BR_1=\lambda_1.$$

Hence $\lambda<\lambda_1<1$ and the $\lambda$ circle lies completely within the $\lambda_1$ circle. From this follows that $|z|>r_1$, and this contradicts the hypothesis. Hence no root of either equation lies within the circle with center at the origin and radius $r_1$. If $|z|=r_1$, then $\lambda=\lambda_1$, and $z$ must lie on the $\lambda_1$ circle; it must therefore be at $R_1$. Hence it must be a root of the second equation only.

Now suppose that $z$ is a root of either equation such that $|z|>r_2$. Then for the circle of the family through $Z$ we have as before $\lambda=|z|^{r+1}/l>{r_2}^{r+1}/l=\lambda_2>1$. The $\lambda_2$ circle encloses the $\lambda$ circle, and hence $|z|<r_2$, a contradiction. If $|z|=r_2$, then $\lambda=\lambda_2$, and $Z$ must be at $R_2$. It has now been proved that all of the roots of either equation lie within the circular ring with the origin as cen-

ter and radii $r_1$ and $r_2$, except the two real roots of the second equation, which lie on the boundaries of the ring.

It has been shown that for any root $z$ of either equation, excepting $r_1$ and $r_2$ we must have $r_1 < |z| < r_2$. Hence if $\lambda$ is the ratio determined by $z$ then

$$r_1^{r+1}/l < |z|^{r+1}/l < r_2^{r+1}/l, \text{ or } \lambda_1 < \lambda < \lambda_2.$$

It then follows that $z$ cannot lie within either the $\lambda_1$ or the $\lambda_2$ circle. It cannot lie on the boundary of either circle, for if it lies, say on the $\lambda_1$ circle, we would have $|z|^{r+1}/l = r_1^{r+1}/l$, or $|z| = r_1$. This is impossible unless $z = r_1$.

3396. [1929, 492]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Let $D$, $E$, $F$ be the feet of the altitudes, and $A'$, $B'$, $C'$, the mid-points of the sides of a triangle $ABC$. Show that the double elements of the three involutions $(BC, DA')$, $(CA, EB')$. $(AB, FC')$ are three pairs of opposite vertices of a complete quadrilateral.

*I. Solution by Mabel M. Young, Wellesley College.*

Let the sides of the triangle $ABC$ meet the sides of the triangle $DEF$ in $O_a$, $O_b$, $O_c$. Since the altitudes are concurrent, $O_a$, $O_b$, $O_c$ are collinear. These points lie on the radical axis of the circumcircle and the nine-point circle of the triangle $ABC$, since each point is the radical centre of these circles and one of the circles having the sides of $ABC$ as diameter. Hence $O_a$, $O_b$, $O_c$ are the centers of the given involutions, which are determined on the sides of the triangle by these two circles. Of the three pairs of double points, either three or one will be real, according as the triangle $ABC$ is acute or obtuse.

Let $A_1A_2$, $B_1B_2$, $C_1C_2$ be the double points of the involutions on $a$, $b$, $c$ respectively, and let the complete quadrilateral which has $A_1A_2$, $C_1C_2$ as two pairs of opposite vertices have $X_1X_2$ as third pair. Two sides of the diagonal triangle of this quadrilateral are $a$ and $c$. The third side, determined by $X_1$ and $X_2$, coincides with $b$, since it cuts $a$ and $c$ in points which are harmonic conjugates of $B$ as to $A_1A_2$ and $C_1C_2$, respectively. The mid-point of segment $X_1X_2$ coincides with $O_b$, since it is collinear with $O_a$ and $O_c$. Vertices $X_1$ and $X_2$ are themselves harmonic conjugates as to $A$ and $C$. Hence $(O_bX_1)^2 = O_bA \cdot O_bC = (O_bB_1)^2$, and $X_1$, $X_2$ coincide with $B_1$, $B_2$.

*II. Solution by A. Pelletier, Montreal, Canada.*

Let $ABC$ be an acute angled triangle. On $BC$ as a diameter describe a circle cutting $AD$ in $G$, and let the tangent at $G$ cut $BC$ in $O$. Then it follows that $A'O \cdot DO = BO \cdot CO = GO^2$, and hence $O$ is the center of the involution determined by the pairs $(BC, DA')$. The circle with $O$ as center and radius $OG$ cuts $BC$ in two points $X, X'$ which are the double points of the involution. We shall suppose that $X$ is the point within the segment $BC$. Similarly, we find $Y$, $Y'$ on $CA$, and $Z$, $Z'$ on $AB$, where again the unaccented letters designate the points

within the segments. Thus $X$, $X'$ divide harmonically $BC$ as well as $A'D$. Since $XGX'$ is a right angle, $GX$ bisects the angle $BGC$; and from this follows that $BX/CX = -BG/GC = -(BD/DC)^{\frac{1}{2}}$. Likewise we find that $CY'/AY' = (CE/EA)^{\frac{1}{2}}$ and $AZ/BZ = -(AF/FB)^{1/2}$. Hence

$$\frac{BX}{CX}\cdot\frac{CY'}{AY'}\cdot\frac{AZ}{BZ} = \left(\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}\right)^{1/2} = 1,$$

since the altitudes meet in a point. Hence $X$, $Y'$, $Z$ lie on a straight line. In a similar manner we show that $Z$, $Y$, $X'$; $X$, $Y$, $Z'$; $Y'$, $X'$, $Z'$ lie in a straight line. Hence the quadrilateral $X$, $Y'$, $X'$, $Y$ has $ZZ'$ for its third diagonal.

*Note by Otto Dunkel.* The above solutions suggest a solution which may be based on the solution of 3387 [1930, 262]. In the latter solution it is shown that the conjugate circle of a triangle is coaxial with the nine-point circle and the circumcircle in all cases. If the triangle $ABC$ has only acute angles, as will be assumed in what follows, then the conjugate circle is imaginary, and it follows that the nine-point circle does not cut the circumcircle in real points. Let $(S)$ denote the circumcircle with the center $S$, and $(N)$ the nine-point circle with the center $N$. The coaxial system of circles determined by these two non-intersecting circles has two limit points $I$ and $J$; or, we may say that $I$ and $J$ are two points on the straight line $SN$ which are conjugate to both $(S)$ and $(N)$. For definiteness, suppose that $I$ is within the two circles. The perpendicular to $IJ$ at its mid-point is the radical axis of the two circles. Let this perpendicular cut $BC$ in $O_x$, and let the circle with center $O_x$ and radius $O_xI$ cut $BC$ in $X$ and $X'$. This circle passes also through $J$, and it is orthogonal to $(S)$ and to $(N)$. Hence $X$ and $X'$ are conjugate to both $(S)$ and $(N)$, and they are in consequence the double points of the involution on $BC$. We shall suppose that $X$ is within the segment $BC$. Then $XIX'$ is a right angle and $XI$ bisects the angle $BIC$. Hence $BX/CX = -BI/CI$. In a similar manner we have

$$CY/AY = -CI/AI,\ AZ/BZ = -AI/BI.$$

From the product of these three equations we infer that $AX$, $BY$, $CZ$ meet in a point. Hence $X'$, $Y'$, $Z'$ lie in a straight line, since these three points are harmonic conjugates of $X$, $Y$, $Z$, respectively, with respect to the corresponding vertices of the triangle. By the use of suitable harmonic pencils it may be shown that $Y', X, Z$; $X', Y, Z$; $Z', Y, X$ lie in straight lines, and the rest easily follows.

Also solved by Paul Wernicke.

3398. [1929, 542]. *Proposed by V. Ivanoff, San Francisco, California.*

Prove that

$$\frac{1}{R^2} = \frac{\sin^6 M}{R_x^2} + \frac{\sin^6 N}{R_y^2} + \frac{\sin^6 P}{R_z^2},$$

where $R$ is the radius of curvature of a given curve in the point $(x_1y_1z_1)$; $R_x$, $R_y$, $R_z$ are the radii of curvature of the projections of this curve in the co-ordinate

planes, $YOZ$, $XOZ$, and $XOY$ in the points $(y_1, z_1)$, $(x_1, z_1)$, and $(x_1, y_1)$, respectively; $M$, $N$, $P$ are the angles between the tangent to the curve in the point $(x_1, y_1, z_1)$ and $OX$, $OY$, and $OZ$, respectively.

*Solution by Robert E. Moritz, University of Washington.*

If we denote the infinitesimal arc of the curve at the point $(x_1, y_1, z_1)$ by $ds$, then obviously $\cos M = dx/ds$, $\cos N = dy/ds$, $\cos P = dz/ds$ and therefore,

$$\sin^2 M = (dy^2 + dz^2)/ds^2,\ \sin^2 N = (dz^2 + dx^2)/ds^2,\ \sin^2 P = (dx^2 + dy^2)/ds^2.$$

Now

$$1/R_x^2 = (dy \cdot d^2z - dz \cdot d^2y)^2/(dy^2 + dz^2)^3;$$

hence

$$(\sin^6 M)/R_x^2 = (dy \cdot d^2z - dz \cdot d^2y)^2/ds^6,$$

and

$$\frac{\sin^6 M}{R_x^2} + \frac{\sin^6 N}{R_x^2} + \frac{\sin^6 P}{R_x^2} = \frac{(dz \cdot d^2y - dy \cdot d^2z)^2 + (dx \cdot d^2z - dz \cdot d^2x)^2 + (dy \cdot d^2x - dx \cdot dy^2)^2}{ds^6},$$

which is the well known expression for $1/R^2$.

Also solved by M. S. Knebelman, J. D. Leith, T. L. Smith, Paul Wernicke, and the Proposer.

3400 [1929, 543]. *Proposed by W. O. Pennell, St. Louis, Mo.*

Find a function of $x$, $f(x)$, such that

$$f(x) = xf(x - r) = x(x - r)f(x - 2r) = x(x - r)(x - 2r)f(x - 3r), \text{ etc.}$$

and $f(r) = r$, where $r$ is a given real quantity $>0$.

*Solution by H. T. Davis, Indiana University.*

The problem is reduced to the solution of the difference equation,

$$u(t + 1) = r(t + 1)u(t),$$

by means of the transformation: $x = r(t+1)$, $u(x) = f(rx)$. Assuming a solution of the form $u(t) = \phi(t)\Gamma(t+1)$, we are led for the definition of $\phi(t)$ to the simple difference equation, $\phi(t+1) = r\phi(t)$. The solution of this equation by ordinary methods is $\phi(t) = \psi(t)r^{t+1}$, where $\psi(t)$ is any periodic function of unit period. Hence the general solution of the original equation will be,

$$f(x) = \psi(xr^{-1})r^{\rho}\Gamma(\rho) \quad \text{where} \quad \rho = xr^{-1} + 1.$$

In order to satisfy the condition $f(r) = r$, it is merely necessary that $\psi(1) = r^{-1}$. A particular solution is given by setting $\psi(xr^{-1}) = r^{-1}$.

The solution just given follows ordinary methods. It might be interesting, therefore, to show how the solution can be attained by a less known method involving the use of differential operators of infinite order.[1]

Using the symbolic operator $f(x-r)=e^{-rz}\to f(x)$, where $z=d/dx$, we may write the original equation in the form,

$$(1 - xe^{-rz}) \to f(x) = 0.$$

Let us seek a solution of the form $f(x)=\int_L e^{xt}v(t)dt$, where $v(t)$ is a function to be determined and $L$ is some path of integration in the complex plane. Introducing this expression into the differential equation we obtain,

$$(1 - xe^{-rz}) \to f(x) = \int_L e^{xt}(1 - xe^{-rt})v(t)dt.$$

Integrating by parts the term multiplied by $x$, we get

$$(1 - xe^{-rz}) \to f(x) = - e^{xt-rt}v(t)\Big|_L + \int_L e^{xt}[v(t) + d\{e^{-rt}v(t)\}/dt]dt.$$

Let us now seek a value of $v(t)$ which satisfies the equation,

$$v(t) + d\{e^{-rt}v(t)\}/dt = 0,$$

and a path $L$ at the extremities of which $e^{t(x-r)}v(t)$ is zero. Since we find $v(t)=Ce^{\mu}$, where $\mu=rt-e^{rt}r^{-1}$, we can choose $L$ to be the path along the real axis from $-\infty$ to $+\infty$. Changing variables from $t$ to $s$ by means of the equation $s=e^{rt}r^{-1}$, $L$ becomes a path from 0 to infinity. We thus obtain,

$$f(x) = C\int_0^{\infty} e^{-s}(rs)^{x/r}ds = Cr^{x/r}\Gamma(xr^{-1} + 1).$$

The path $L=\infty$ to 0, $n$ loops about the origin, 0 to $\infty$ would give:

$$C(e^{2\pi inx/r} - 1)r^{x/r}\Gamma(xr^{-1} + 1).$$

The desired function which reduces to $r$ for $x=r$ is obtained by letting $C=1$ in the less general preceding solution.

Also solved by G. D. Leith, Byron T. Roberts, T. L. Smith, and the Proposer.

3401 [1929, 543]. *Proposed by Paul Wernicke, Washington, D. C.*

Let $a$, $b$, $c$, $d$, be four lines in a plane no three of which are concurrent. Let

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[1] See for example: T. Lalesco, *Sur l'équation de Volterra*, Journal de Mathématiques, (6), vol. 4, (1908), p. 194, and E. Hilb, *Lineare Differentialgleichungen unendlicher Ordnung*, Mathematische Annalen, vol. 82 (1921), p. 20.

$f$ be the join of the intersections $ac$ and $bd$ and $g$ the join of the intersections of $ad$ and $bc$. Prove that

$$(\sin af/\sin fc)/(\sin ag/\sin gd) = (\sin bg/\sin gc)/(\sin bf/\sin fd).$$

*I. Solution by Robert E. Moritz, University of Washington.*

Let us first assume that no two of the given lines are parallel. Then any three of the six lines, $a, b, c, d, f, g$, which are not concurrent, will form a triangle. Let us denote the sides of any such triangle, say the triangle formed by the lines $b, c, d$, by $\bar{b}, \bar{c}, \bar{d}$, and the area of the triangle by $T(b, c, d)$. With this notation we have

$$2 \cdot T(a, f, d) = \bar{a} \cdot \bar{f} \cdot \sin af = \bar{f} \cdot \bar{d} \cdot \sin fd, \tag{1}$$

$$2 \cdot T(b, f, c) = \bar{f} \cdot \bar{c} \cdot \sin fc = \bar{b} \cdot \bar{f} \cdot \sin bf, \tag{2}$$

$$2 \cdot T(a, g, c) = \bar{a} \cdot \bar{g} \cdot \sin ag = \bar{g} \cdot \bar{c} \cdot \sin gc, \tag{3}$$

$$2 \cdot T(b, g, d) = \bar{g} \cdot \bar{d} \cdot \sin gd = \bar{b} \cdot \bar{g} \cdot \sin bg. \tag{4}$$

If now we divide the quotient of (1) divided by (2) by the quotient of (3) divided by (4), and divide out the common factor $\bar{d}/\bar{c}$, we obtain the relation in question.

The relation just established for the cases when no two of the given lines are parallel will continue to hold true when two or more of the given lines are parallel, for every such case may be considered the limiting case of a quadrilateral for which the relation is invariant. It must then hold true in the limit.

*II. Solution by S. F. Bibb, Armour Institute of Technology.*

The following is a general statement of the problem and its solution: Suppose that the $n$ sides of a polygon are $a_1, a_2, a_3, \cdots a_n$. Let $b_1$ be a diagonal drawn from the intersection of $a_n a_1$ to the intersection of $a_2 a_3$, and $b_2$ a diagonal drawn from the intersection of $a_1 a_2$ to the intersection of $a_3 a_4$, and $b_3$ a diagonal drawn from the intersection of $a_2 a_3$ to the intersection of $a_4 a_5$, and similarly for $b_4, b_5, \cdots b_n$. Prove that

$$\frac{\sin a_1b_1 \sin a_2b_2 \sin a_3b_3 \cdots \sin a_nb_n}{\sin a_2b_1 \sin a_3b_2 \sin a_4b_3 \cdots \sin a_nb_{n-1} \sin a_1b_n} = (-1)^n.$$

*Proof*: By law of sines one obtains:

$$\begin{aligned}
(\sin a_1b_1)/(\sin a_2b_1) &= -(a_2)/(a_1),\\
(\sin a_2b_2)/(\sin a_3b_2) &= -(a_3)/(a_2),\\
(\sin a_3b_3)/(\sin a_4b_3) &= -(a_4)/(a_3)\\
&\cdots\cdots\cdots\cdots,\\
&\cdots\cdots\cdots\cdots\\
(\sin a_nb_n)/(\sin a_1b_n) &= -(a_1)/(a_n).
\end{aligned}$$

Multiplying these equations together one gets the result as stated. For $n=4$, this gives the solution of the proposed problem.

Also solved by E. M. Berry, W. E. Buker, Solomon Kullback, J. D. Leith. A. Pelletier, T. L. Smith, Abraham Sinkov, and the Proposer.

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## NOTES AND NEWS

*Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.*

Professor Enrico Bompiani, of the University of Rome, has recently delivered the following lectures as Visiting Professor of the American Mathematical Society: "Differential Equations and Projective Geometry," at Columbia University on May 16; "A Contribution to the Geometry of Paths," at Princeton University on May 19; "Projective Differential Geometry and the Italian School" (two lectures), at Cornell University on May 21, 22; "What is Geometry," at the Ohio State University on May 28; "Italian Contributions to Modern Mathematics," at the University of Iowa on June 30; "What is Geometry" and "Italian Contributions to Modern Mathematics," at the University of Chicago on July 1, 8. During the present Summer Quarter, Professor Bompiani is lecturing at the University of Chicago on "Analytic Projective Geometry" and on "Projective Differential Geometry of Hyperspaces." From October 5 to November 12 he will lecture at Johns Hopkins University on "The Differential Equations of Projective Geometry."

Professors P. W. Bridgman, of Harvard University, and Stephen Timoshenko, of the University of Michigan, are among the members of the newly formed scientific advisory board of the Westinghouse Research Laboratories.

Dr. Karl Menger, of the University of Vienna, has been appointed lecturer in mathematics at Harvard University for the first half of the year 1930–31.

Dr. J. A. Schouten will deliver a series of informal lectures at Harvard Unisity in the fall of 1930.

Dr. J. Frenkel, professor of theoretical physics at the Polytechnic Institute of Leningrad, will lecture on wave mechanics and conduct a seminar on problems in modern physics at the University of Minnesota during the academic year 1930–31.

Dr. Joseph Eugene Rowe, President of Clarkson Memorial College of Technology, was granted the honorary degree of LL.D. at the annual commencement of Gettysburg College held on June 9, 1930.

The following mathematicians have been awarded Guggenheim fellowships for 1930: Professor A. N. Gándara, of the National Preparatory School of

Mexico, for studies in differential geometry and harmonic analysis at the Massachusetts Institute of Technology; Professor H. S. Vandiver, for research in Europe on Fermat's last theorem, laws of reciprocity, and related topics in the theory of algebraic numbers.

Professor C. L. Arnold, of the Ohio State University, has retired from active teaching. He was given the rank of professor emeritus.

Dr. J. Hobert Bushey has been appointed to an assistant professorship in mathematics at Hunter College of the City of New York.

Assistant Professor W. F. Cheney, of Tufts College, has been appointed head of the department of mathematics at the Connecticut Agricultural College.

Dr. Alonzo Church has been appointed assistant professor of mathematics at Princeton University.

Dr. T. F. Cope has been appointed assistant professor and head of the department of mathematics at Marietta College.

Associate Professor Louise D. Cummings, of Vassar College, has been promoted to a professorship of mathematics.

Dr. H. B. Curry has been appointed assistant professor of mathematics at Pennsylvania State College.

Dr. Julia Dale, head of the department of mathematics at Delta State College in Mississippi, has been appointed assistant professor of mathematics at Duke University.

Dr. Marguerite D. Darkow has been promoted to an assistant professorship in mathematics at Hunter College of the City of New York.

Dr. Jesse Douglas has been appointed assistant professor of mathematics at the Massachusetts Institute of Technology.

Mr. R. D. Douglass, of the Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematcs.

Professor W. E. Edington, of Purdue University, has been appointed professor and head of the department of mathematics at DePauw University.

Assistant Professor Philip Franklin, of the Massachusetts Institute of Technology, has been promoted to an associate professorship of mathematics.

Dr. B. P. Gill has been promoted to an assistant professorship of mathematics at the College of the City of New York.

Dr. E. L. Hill has been appointed assistant professor of theoretical physics at the University of Minnesota.

Assistant Professor B. P. Hoover, of the Carnegie Institute of Technology, has been promoted to an associate professorship of mathematics.

Dr. Jewell C. Hughes has been appointed to an assistant professorship in mathematics at Hunter College of the City of New York.

Assistant Professor R. P. Johnson, of the Carnegie Institute of Technology, has been promoted to an associate professorship of mathematics.

Assistant Professor R. B. Lindsay, of Yale University, has been appointed associate professor of theoretical physics at Brown University.

Associate Professor P. H. Linehan, of the College of the City of New York, has been promoted to a professorship of mathematics.

Dr. W. H. McEwen, of the University of Minnesota, has been appointed assistant professor of mathematics in Mount Allison University, Sackville, N. B.

Associate Professor M. L. MacQueen, of Southwestern University, has been promoted to a professorship of mathematics.

Assistant Professor Florence M. Mears, of Pennsylvania State College, has been appointed assistant professor of mathematics at George Washington University.

Dr. T. W. Moore has been appointed assistant professor of mathematics at Indiana University.

Associate Professor Marston Morse has been promoted to a professorship at Harvard University.

Associate Professor E. J. Oglesby, of Washington Square College, New York University, has been promoted to a professorship of mathematics.

Dr. G. A. Parkinson has been promoted to an associate professorship of mathematics in the extension division of the University of Wisconsin at Milwaukee.

Professor H. B. Phillips, of the Massachusetts Institute of Technology, has been granted leave of absence for the academic year 1930–31.

Assistant Professor Emory Potterstarke has been promoted to an associate professorship of mathematics at Rutgers University.

Dr. I. I. Rabi has been promoted to an assistant professorship of theoretical physics at Columbia University.

Mr. L. H. Rice, of the Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematics.

Dr. J. H. Roberts has been appointed adjunct professor of mathematics at the University of Texas.

Assistant Professor George Rutledge, of the Massachusetts Institute of Technology, has been promoted to an associate professorship of mathematics.

Mr. George Sauté has been appointed assistant professor of mathematics at Cleveland College.

Mr. C. N. Shuster has been promoted to an assistant professorship of mathematics at the State Teachers College at Trenton, N. J.

Dr. C. H. Smiley has been appointed assistant professor of mathematics at Brown University.

Mr. I. S. Sokolnikoff has been promoted to an assistant professorship of mathematics at the University of Wisconsin.

Dr. E. H. Taylor has been promoted to a professorship of mathematics at the Eastern Illinois State Normal College.

Assistant Professor R. W. Veatch, of Ursinus College, has been appointed assistant professor of mathematics at the University of Tulsa.

Assistant Professor J. L. Walsh has been promoted to an associate professorship of mathematics at Harvard University.

Dr. Dorothy W. Weeks has been appointed professor of physics at Wilson College.

Assistant Professor E. A. Whitman, of the Carnegie Institute of Technology, has been promoted to an associate professorship of mathematics.

Associate Professor D. V. Widder, of Bryn Mawr College, has been promoted to a professorship of mathematics.

Professor A. R. Wilson, of Haverford College, delivered a lecture on March 27 for the mathematics clubs of Rutgers University on the subject, "Space Filling Polyhedra."

Mr. Carleton R. Worth, of Rutgers University, has been granted leave of absence for the year 1930–31. He will study at the California Institute of Technology.

Associate Professor Mabel M. Young, of Wellesley College, has been promoted to a professorship of mathematics.

At the University of Chicago, Mr. Lawrence M. Graves and Mrs. Mayme I. Logsdon have been promoted to associate professorships in mathematics. Mr. Ralph G. Sanger and Mr. Clifford W. Mendel are appointed to instructorships, Mr. Ralph E. Huston to an assistantship, and Mr. Max Coral and Mr. Arnold E. Ross to research assistantships for 1930–31.

The following appointments to instructorships are announced:

University of British Columbia, Mr. F. J. Brand.

Duke University, Mr. F. G. Dressel of the University of Michigan and Mr. J. A. Greenwood of the University of Missouri.

University of Iowa, Dr. J. M. Earl.

Massachusetts Institute of Technology, Part time: Mr. H. A. Giddings of the versity of Vermont, Mr. E. M. Pease of Purdue University, Mr. J. G. Estes of the Texas Christian University. Full time: Mr. S. B. Littauer.

North Carolina State College, Mr. E. R. Elliott.

Princeton University, Dr. K. E. Rosinger (in the department of Philosophy.)

Rutgers University, Mr. H. B. Huntley, and Mr. H. S. Grant of the University of Buffalo.

Dr. T. E. McKinney, for twenty years professor of mathematics at the University of South Dakota, died on April 14, 1930, at the age of sixty-six.

Dr. J. L. Markley, professor emeritus of mathematics at the University of Michigan, died on April 20, 1930, in his seventy-first year.

Dr. W. E. Story, professor emeritus of mathematics at Clark University, died on April 11, 1930, at the age of seventy-nine.

Professor Florian Cajori died suddenly of pneumonia on August 14, 1930, at his home in Berkeley, California. He was a charter member of the Mathematical Association of America and was one of an original group of four (later enlarged to twelve) representatives of mid-western universities and colleges who made possible the re-establishment of the American Mathematical Monthly on a sound financial basis. A detailed account of his historical researches will be published in the *Monthly* in due course.

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[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

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## DIRECTORY

**EDITORIAL CORRESPONDENCE** should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

**BOOKS FOR REVIEW** should be sent to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

**BUSINESS CORRESPONDENCE** should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

### MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.
INDIANA, Earlham College, May 2-3.
IOWA, Ames, Iowa, May 2-3.
KANSAS, February 15.
KENTUCKY, Lexington, Ky., April 5.
LOUISIANA-MISSISSIPPI, Cleveland, Miss., March 7-8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA. May 10
MICHIGAN, Ann Arbor, Mich., March 22.
MINNESOTA, Carleton College, May 17.
MISSOURI.
NEBRASKA, Peru, Neb., May 9.
OHIO, Columbus, Ohio, April 3.
PHILADELPHIA, Philadelphia, Pa., November 29.
ROCKY MOUNTAIN, Denver, Colo., April 11-12.
SOUTHEASTERN, Atlanta, Ga., May 2-3.
SOUTHERN CALIFORNIA, University of Southern California, Los Angeles, Calif., March 8.
TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

## THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The fourteenth annual meeting of the Rocky Mountain Section of the Mathematical Association was held at the University of Denver, Denver, Colorado, on April 11–12, 1930. There were three sessions, Professor G. W. Gorrell acting as chairman at each.

The attendance was forty-two including the following twenty-seven members of the association: C. F. Barr, J. Britton, A. G. Clark, J. R. Everett, J. C. Fitterer, G. W. Gorrell, S. G. Hacker, C. A. Hutchinson, D. Jackson, H. Karnow, A. J. Kempner, Miss Claribel Kendall, A. J. Lewis, G. H. Light, A. S. McMaster, J. Q. McNatt, W. K. Nelson, Miss Greta Neubauer, Miss L. R. Odell, E. J. Purcell, E. D. Rainville, A. W. Recht, W. J. Risley, L. J. Rote, Miss Mary Sabin, C. H. Sisam, Miss Adela M. Thom.

The following officers were elected for the coming year: Professor Claribel Kendall, University of Colorado; Vice-Chairman Professor C. F. Barr, University of Wyoming.

The following papers were read:

1. "Fregier's theorem" by Professor Francis Regan, Colorado Agricultural College, by invitation.
2. "Predicting occultations" by Professor A. W. Recht, University of Denver.
3. "Foci of algebraic curves" by Professor Claribel Kendall, University of Colorado.
4. "On the invariance of certain types of areas" by Professor A. G. Clark, Colorado Agricultural College.
5. "A problem in partial correlation" by Professor G. H. Light, University of Colorado.
6. "Focal surface of a normal congruence of an ellipsoid" by Professor J. R. Everett, Colorado School of Mines.
7. "A formula in terms of greatest integers giving parcel post charges as a function of weight and distance" by Professor W. K. Nelson, University of Colorado.
8. "The elliptic modular group and applications to the theory of functions" by Mr. Earl Rainville, University of Colorado.
9. "Theory of numbers and the multiplication table" by Professor A. J. Kempner, University of Colorado.
10. "Formulas of correlation in several variables" by Professor Dunham Jackson, University of Minnesota.

Abstracts of these papers follow:

1. Mr. Regan presented Fregier's theorem: if a variable chord $PQ$ of a conic subtends a right angle at any fixed point $V$ on the conic it passes through a fixed point $F$ which lies on the normal to the conic at $V$. The proofs for the parabola, ellipse, and hyperbola were given. The theorems dealing with the locus of the Fregier points of each conic were developed, and several corollaries